SOLUTIONS

CAMBRIDGE PROBLEMS,

PROPOSED BY THE MODERATORS

TO THE CANDIDATES FOR HONORS

AT THE EXAMINATION FOR THE DEGREE OF B.A.

JANUARY, 1831.

TO WHICH ARE ADDED,

ESSAYS ON VARIOUS POINTS OF

PURE AND MIXED MATHEMATICS.

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ADVERTISEMENT.

THE design of this Work being not merely to present the Student with the formal Solutions of the Senate-House Problems; but to facilitate his application of general principles to the solutions of others (to which purpose the Senate-House Problems are in general well adapted); the Author considers this a sufficient reason for having dwelt at some length on those points often imperfectly considered by the Student; to the exclusion of some of the simpler details in which little or no assistance is required.

In explaining the *Undulatory Theory of Light*, the Author feels himself bound to acknowledge having made an extensive use of the writings of Mr. Fresnel: he also takes this opportunity of acknowledging the kindness of Mr. Bowstead and Mr. Challis in allowing the use of their Problems, to the latter of which gentlemen he is indebted for Articles 13, 19, and 20, of the Friday Evening Problems.

SOLUTIONS

CAMBRIDGE PROBLEMS.

JANUARY 1831.

MONDAY MORNING.

1. IF n be any whole number, one of the three n^2, $n^2 + 1$, $n^2 + 4$, is divisible by 5 without remainder.

Since n is always of one of the forms

$$5m, \ 5m + 1, \ 5m + 2, \ 5m + 3, \ 5m + 4;$$

n^2 is always of one of the forms

$(5m)^2 \qquad$ which is of the form $\qquad 5m$

$(5m + 1)^2 \ \dots\dots\dots\dots\dots \ 5m + 1$

$(5m + 2)^2 \ \dots\dots\dots\dots\dots \ 5m + 4$

$(5m + 3)^2 \ \dots\dots\dots\dots\dots \ 5m + 4$

therefore each of these squares will be divisible by 5 when the second has been increased by 4, and the two last by 1; whence the truth of the proposition.

Q. E. D.

A

2. A debt of $a \pounds$. accumulating at compound interest, is discharged in n years by annual payments of $\frac{a}{m} \pounds$; prove that $n = -\dfrac{\log(1 - mr)}{\log(1 + r)}$, r being the interest of $1 \pounds$. for one year.

In Wood's Algebra (Art. 402) we have the formula

$$P R^n = \frac{R^n - 1}{R - 1} \times A,$$

which is adapted to this problem by making

$$P = a, \ R = 1 + r, \text{ and } A = \frac{a}{m}:$$

whence $R^n = \dfrac{A}{A + (1 - R) P}$,

or $\quad (1 + r)^n = \dfrac{1}{1 - mr}$;

$$\therefore n = -\frac{\log(1 - mr)}{\log(1 + r)}.$$

Q. E. D.

3. If α be less than $45°$, shew that

$$2 \sin\left(\frac{\pi}{4} \pm \alpha\right) = \sqrt{(1 + \cos 2\alpha)} \pm \sqrt{(1 - \cos 2\alpha)},$$

and $2 \cos\left(\dfrac{\pi}{4} \pm \alpha\right) = \sqrt{(1 + \cos 2\alpha)} \mp \sqrt{(1 - \cos 2\alpha)}.$

$$\sin\left(\frac{\pi}{4} \pm \alpha\right) = \sin\frac{\pi}{4} . \cos \alpha \pm \cos\frac{\pi}{4} . \sin \alpha$$

$$= \frac{\cos \alpha \pm \sin \alpha}{\sqrt{2}},$$

$$\left(\text{since } \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt 2} \right);$$

$$\therefore \text{ since } \cos a = \frac{\sqrt{(1 + \cos 2 a)}}{\sqrt 2}$$

$$\text{and } \sin a = \frac{\sqrt{(1 - \cos 2 a)}}{\sqrt 2}$$

$$2 \sin\left(\frac{\pi}{4} \pm a \right) = \sqrt{(1 + \cos 2 a)} \pm \sqrt{(1 - \cos 2 a)}.$$

$$\text{Also, } \cos\left(\frac{\pi}{4} \pm a \right) = \cos\frac{\pi}{4} \cdot \cos a \mp \sin\frac{\pi}{4} \cdot \cos a$$

$$= \frac{\cos a \mp \sin a}{\sqrt 2};$$

$$\therefore 2 \cos\left(\frac{\pi}{4} + a \right) = \sqrt{(1 + \cos 2 a)} \mp \sqrt{(1 - \cos 2 a)},$$

The two terms of the right-hand members of the two equations thus found retain their signs as above, as long as a is less than 45°, but become, one or both of them, affected with contrary signs in the contrary case. The apparent exception to the generality of these formulæ arises from the ambiguity in sign of those terms obtained by the extraction of the square root.

4. Prove that the locus of the points of bisection of any number of chords to an ellipse, which pass through the same point, is an ellipse, and find the magnitude and position of the axes, when the coordinates to the point are given.

Let C be the centre of the given ellipse, A the given point, from which draw the straight line APQ cutting the ellipse in the points P, Q; bisect PQ in N, join CN and produce it to meet the ellipse in M.

Draw the ordinates $Nn = y$, $Mm = Y$ to the major axis, and let $Cn = x$, $CN = X$; let a, β be the co-ordinates of the given point A, and let the equation to the line $APNQ$ be

$$y' - \beta = m (x' - a):$$

also, if a, b be the semi-axes of the ellipse, we shall have

$$Y^2 = \frac{b^2}{a^2} (a^2 - X^2).$$

Now, a tangent to the ellipse at M will be parallel to APQ;

$$\therefore \; m = \frac{dY}{dX} = -\frac{b^2}{a^2} \cdot \frac{X}{Y} = -\frac{b^2}{a^2} \cdot \frac{x}{y};$$

$$\left(\text{Since, by the similar triangles } CNn, \, CMm, \frac{x}{y} = \frac{X}{Y} : \right)$$

The equation of $APNQ$ becomes, therefore,

$$y' - \beta = -\frac{b^2}{a^2} \cdot \frac{x}{y} (x' - a).$$

Now, to limit the values of x', y', in this equation, to the point N of bisection of PNQ, we must make $x' = x$ and $y' = y$, and it will then become, after reduction,

$$a^2 y^2 - a^2 \beta y + b^2 x^2 - b^2 a x = 0;$$

$$\text{whence } y = \frac{\beta}{2} \pm \sqrt{\left\{ \frac{\beta^2}{4} + \frac{b^2}{a^2} (ax - x^2) \right\}},$$

which is manifestly the equation to an ellipse, similar to, and having its axes parallel to, those of the given ellipse. In the equation last found the rational part expresses the position of the axis that bisects all the chords parallel to the axis of y, the half of any one of which is expressed by the irrational part. Since, then, these chords are equal to nothing only at the extremities of this axis; the difference of the two values of x which make $\frac{\beta^2}{4} + \frac{b^2}{a^2}(ax - x^2) = 0$, must be equal to the length of this axis. By solving the above equation with respect to

x, and finding the values of y which make the irrational part of it $= 0$, we similarly obtain the length of the other axis. The axes are thus found to be respectively,

$$\frac{1}{b} \, \sqrt{(a^2 b^2 + a^2 \beta^2)}$$

$$\text{and} \frac{1}{a} \, \sqrt{(a^2 b^2 + a^2 \beta^2)}.$$

5.   Shew that three conditions must be satisfied by the constant co-efficients in the general equation of lines of the second order, that it may be the equation of two straight lines.

If $\begin{array}{l} Ay + B'x + C = 0, \\ A'y + B'x + C' = 0, \end{array} \Big\}$ be the equations of two straight lines, in which A, B, C, A', B', C' are always possible and rational, their product necessarily gives an equation of the second order, in x and y, whose roots are always rational and possible whatever be the value of x or y. Every such equation, therefore, that represents two straight lines, must arise from the product of two simple equations whose terms are possible and rational.   Now the general equation

$$a y^2 + b \, x y + c \, x^2 + d y + e x + f = 0,$$

gives

$$y = -\frac{b \, x + d}{2 \, a} \pm \sqrt{\big\{ (b^2 - 4 \, a \, c) \, x^2 + 2 \, (b \, d - 2 \, a \, e) \, x + d^2 - 4 \, a f \big\}},$$

$$x = -\frac{b \, y + e}{2 \, c} \pm \sqrt{\big\{ (b^2 - 4 \, a \, c) \, x^2 + 2 \, (b \, e - 2 \, c \, d) + e^2 - 4 \, c f \big\}},$$

which expressions, in the case supposed above, must respectively be possible and rational for every value of x and y: whence, the part under the radical sign in each must be a square; which gives the two conditions

$$(b^2 - 4 \, a \, c)(d^2 - 4 \, a f) = (b \, d - 2 \, a \, e)^2 \qquad (1)$$
$$(b^2 - 4 \, a \, c)(e^2 - 4 \, c f) = (b \, e - 2 \, c \, d). \qquad (2)$$

Also, when the three terms under either radical sign are those of a square, $b^2 - 4\,a\,c$ must be positive,

or $\qquad b^2 > 4\,a\,c.$ $\qquad\qquad\qquad$ (3)

We might observe that $d^2 - 4\,a f$ and $e^2 - 4\,c f$ must likewise be positive; but these conditions are already expressed by the equations (1) and (2); for as the right-hand members are essentially positive, the two factors on the left hand of each must necessarily have the same sign.

6.　If P, Q be two points in a curve referred to a pole S, and PM, QN perpendiculars on a fixed straight line TMN, find the equation of the curve, when the area SPQ is to the area $PQMN$ as 1 to n, and construct the curve when $n = 1$.

From S let fall the perpendicular $SA = b$ on the fixed line TMN; and first, let S be the origin of rectangular co-ordinates, and SA, indefinitely produced, the axis of y.

The area SPQ is then $\frac{1}{2}\int (y\,dx - x\,dy)$; but when the origin is transferred to A, $SPQ = \frac{1}{2}\int \left\{(y + b)\,dx - x\,dy\right\}$, and the area $PQMN = \int y\,dx$:

$$\therefore (y + b)\,dx - x\,dy : 2\,y\,dx :: 1 : n,$$

$$\text{or } \left\{(n - 2)\,y + n\,b\right\}\,dx = x\,dy;$$

$\therefore$ by integration $\left\{(n - 2)\,y + n\,b\right\}^{\frac{n}{n-2}} = a^{\frac{2}{n-2}}\,x$;

$a^{\frac{2}{n-2}}$ being an arbitrary constant of such a form that a may represent the magnitude of a line.

When $n = 1$, the equation becomes

$$y = b - \frac{a^2}{x},$$

which is that of an equilateral hyperbola whose asymptote parallel to the axis of x is at a distance $= 2b$ from S; its concavity being turned towards the axis of x.

7. Trace the curve of which $a^m y = x(x-a)^m$ is the equation, m being an even number; find its maximum and minimum ordinates, point of contrary flexure, and the angle at which it cuts the axis of x.

Since y vanishes when $x = 0$, and $x = a$, and for no other values of x; and since m is even, y is positive when x is positive, and negative when x is negative: therefore the curve only cuts the axes at the origin, and extends infinitely on the positive and negative sides of the axes of x and y.

Since $a^m \dfrac{dy}{dx} = (x-a)^m + m x (x-a)^{m-1}$,

$$a^m \frac{d^2 y}{dx^2} = 2m(x-a)^{m-1} + m(m-1)x(x-a)^{m-2};$$

$$a^m \frac{d^m y}{dx^m} = m^2.(m-1)\ldots3.2(x-a) + m(m-1)\ldots3.2.1;$$

when $x = 0$, $\dfrac{dy}{dx} = 1$; $\therefore$ the curve cuts the axes of x and y at an angle of $45°$: but when $x = a$, $\dfrac{dy}{dx} = 0$; $\therefore$ the curve touches the axis of x at this point.

Again, $\dfrac{dy}{dx} = 0$, when $x = \dfrac{a}{m+1}$, which gives the position of the maximum ordinate between the points where $x = 0$ and $x = a$; at which last the ordinate vanishes and is a minimum, because it again increases when $x > a$.

When $\dfrac{d^2 y}{dx^2} = 0$, we have $x = a$, and

$$x = \frac{2a}{m+1},$$ the last of which values only belongs

to a point of contrary flexure, since it does not make $\dfrac{d^3 y}{d x^3}$ vanish at the same time with $\dfrac{d^2 y}{d x^2}$.

Again, since m is even, when x is positive and less than $\dfrac{2 a}{m + 1}$, $\dfrac{d^2 y}{d x^2}$ is negative; and when x is greater than $\dfrac{2 a}{m + 1}$, $\dfrac{d^2 y}{d x^2}$ is always positive: it vanishes when $x = a$, but does not change its sign at this point: we also observe that all the differential co-efficients up to the order $m - 1$ vanish at this point; which is, therefore, a point of *multiple undulation*. This arises from the curve being a simplification of some more complicated curve from whose equation several constants have been made to disappear, by which several points of undulation have been made to unite in one.

Lastly, when y is negative, $\dfrac{d^2 y}{d x^2}$ is always negative, and therefore the part of the curve on the negative side of the axis of x is every where convex towards it.

As the curve consists of but one continuous line, its form is readily traced from the above description.

8. Required the equation of the plane in which two given straight lines lie, which intersect each other in space.

Let the equations of the given lines be

$$\left. \begin{array}{l} x = m z + a \\ y = n z + b \end{array} \right\} \quad (1)$$

$$\left. \begin{array}{l} x = \mu z + \alpha \\ y = \nu z + \beta \end{array} \right\} \quad (2)$$

that of the plane $z = A x + B y + C$; in which A, B, C are to

be determined. The conditions will evidently be fulfilled by subjecting the plane to pass through any two points in each of the lines (1) and (2); and to effect this most simply we shall suppose the plane to pass through the intersection of those two lines, and through each of the points in which they meet the plane of xy; that is, (if $x = p$, $y = q$, $z = r$ be the co-ordinates of the point of intersection of these lines,) through three points in which the values of x, y, z are respectively,

$$(a, b, 0), \ (\alpha, \beta, 0), \ (p, q, r).$$

We thus have the following equations for determining A, B, C.

$$r = A p + B q + C$$
$$0 = A a + B b + C$$
$$0 = A \alpha + B \beta + C;$$

$$\therefore A = - \frac{(b - \beta) r}{(a - \alpha)(q - b) - (b - \beta)(p - a)}$$

$$B = \frac{(a - \alpha) r}{(a - \alpha)(q - b) - (b - \beta)(p - a)}$$

$$C = \frac{a b - \alpha \beta}{(a - \alpha)(q - b) - (b - \beta)(p - a)},$$

or, since $p - a = m r$, and $q - b = n r$,

$$A = - \frac{b - \beta}{n(a - \alpha) - m(b - \beta)}$$

$$B = \frac{a - \alpha}{n(a - \alpha) - m(b - \beta)}$$

$$C = \frac{a b - \alpha \beta}{n(a - \alpha) - m(b - \beta)};$$

$\therefore$ the equation of the plane is

$$\{ n(a - \alpha) - m(b - \beta) \} z = (\beta - b)x + (a - \alpha)y + a b - \alpha \beta.$$

The relation that must subsist between the given quantities that the lines may intersect each other is found by eliminating

x and y from the four equations (1) (2), and equating the two values of z so found : this gives the equation of condition

$$\frac{a - a}{m - \mu} = \frac{\beta - b}{n - \nu}.$$

9. Three uniform beams AB, BC, CD, of the same thickness, and of lengths l, $2\,l$, l respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius of which $= 2\,l$, so that the middle point of BC, and the extremities AD are in contact with the sphere : shew that the pressure at the middle point of $BC = \dfrac{91}{100}$ of the weight of the beams.

Let O be the centre of the sphere, and $\delta =$ the angle made by AO with the horizon; let R be the pressure at the middle point of BC, and R' that at A or D.

By the principles of equilibrium, the whole weight of the beams must be equal to the sum of the vertical parts of their several pressures on the sphere; and therefore as these pressures are all towards the centre, we have the equation

$$4\,l = R + 2\,R' \sin \delta.$$

To get another equation, we may consider that the sum of the moments of the forces acting on AB, about the point B, must $= 0$. These forces are gravity (represented by l) acting vertically at the middle point, and the reaction R' acting perpendicularly at A :

$$\therefore 2\,R' = l \sin \delta\,;$$

$\therefore$ by substitution, $4\,l = R + l \sin^2 \delta\,;$

$$\therefore R = \left(1 - \frac{1}{4} \cdot \sin^2 \delta\right) 4\, l$$

$$= \left(1 - \frac{1}{4} \cdot \sin^2 \delta\right) \cdot \text{weight of beams.}$$

But if $\angle\ AOB = a,\ \ \delta = \frac{\pi}{2} - 2\,a,\ $ and $\tan a = \frac{AB}{AO} = \frac{1}{2}$;

$$\therefore \sin^2 \delta = \cos^2 2\,a = \frac{9}{25};$$

$$\therefore 1 - \frac{1}{4}\sin^2 \delta = \frac{91}{100};$$

$$\therefore R = \frac{91}{100} \cdot \text{weight of beams.}$$

<div align="right">Q. E. D.</div>

10. Having given the latitudes of two places on the Earth's surface, one of which is N. E. of the other, find the difference of their longitudes and their distance from each other, considering the Earth a sphere.

Let the points P, Q denote the positions of the two places, of which P is N. E. of Q, and N that of the north pole : then, joining by great circles, NP, PQ, QN, we have given in the triangle NPQ, the co-latitudes $NP = 90° - \lambda$, $NQ = 90° - \lambda'$, and the $\angle\ PQN = 45°$; to find the $\angle\ PQN = $ difference of longitudes $= l$, and the distance $PQ = \delta$.

Then, taking the auxiliary $\angle\ NPQ = \theta$, we have

$$\sin \theta = \frac{\cos \lambda' \cdot \sin 45°}{\cos \lambda}; \ \text{ then, by Napier's analogies,}$$

$$\cot \frac{l}{2} = \tan \frac{1}{2} (\theta + 45°) \cdot \frac{\sin \frac{1}{2} (\lambda + \lambda')}{\cos \frac{1}{2} (\lambda - \lambda')},$$

$$\tan \frac{\partial}{2} = \cot \frac{1}{2} \left(\lambda + \lambda' \right) \frac{\cos \frac{1}{2} \left(\theta + 45° \right)}{\cos \frac{1}{2} \left(\theta - 45° \right)}.$$

11.   If a body be attracted by a force which varies as $\frac{1}{(\text{dist}^n)}$, find the value of n when the velocity acquired from an infinite distance to a distance r from the centre, is equal to the velocity that would be acquired from r to $\frac{r}{4}$.

If v be the velocity, and P the force at distance r from the centre, we have (Whewell's Dynamics, p. 26.) $v^2 = C - 2f P d r$.

But if $m =$ force at distance $= 1$, $P = \frac{m}{r^n}$;

$$\therefore v^2 = C - 2 m \int \frac{d r}{r^n}$$

$$= C + \frac{2 m}{n - 1} \cdot \frac{1}{r^{n-1}};$$

But if R be the distance from the centre at which the body begins to move,

$$0 = C + \frac{2 m}{n - 1} \cdot \frac{1}{R^{n-1}}; \ \therefore v^2 = \frac{2 m}{n - 1} \left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}} \right),$$

which, when R is infinite, becomes $v^2 = \frac{2 m}{n - 1} \cdot \frac{1}{r^{n-1}}$.

Again, changing R into r and r into $\frac{r}{4}$ in the general expression, it becomes $v^2 = \frac{2 m}{n - 1} \left(\frac{4^{n-1}}{r^{n-1}} - \frac{1}{r^{n-1}} \right)$, which, by the problem, is to be equal to the expression last found:

$$\therefore \; \frac{1}{r^{n-1}} = \frac{4^{n-1} - 1}{r^{n-1}}, \; \text{ or } 2^{2\,n-2} = 2\,;$$

$$\therefore \; 2\,n - 2 = 1, \; \therefore \; n = \frac{3}{2}.$$

12. Water retained at a constant elevation, issues from one vessel into another, through a cylindrical pipe of radius r, and thence into the air through another cylindrical pipe of radius r'; having given the depth **H** of the mouth of the latter below the constant surface, find the velocity and pressure at a given depth h in the other pipe.

If p be the pressure at any point of a fluid in motion where the velocity is v, and P the pressure at the same point which would result from the impressed forces only, if the fluid were at rest, we have *(Moseley's Hydrostatics, Art.* 132.)

$$p = P \mp D\,v^2 + C;$$

D being the density of the fluid which we shall suppose unity. Now if z be the depth of any point in the fluid below the surface, and Π the atmospheric pressure on the surface; this equation becomes

$$p = g\,z - \frac{v^2}{2} + \Pi\,;$$

$\therefore$ if v' be the velocity at the orifice; (where p is again equal to Π) we have $v'^2 = 2\,g\,H$.

Also, at a depth h, $p = g\,h - \dfrac{v^2}{2} + \Pi$.

But, since the velocities are inversely as the sections,

$$v = v' \cdot \frac{\pi\,r'^2}{\pi\,r^2} = v' \cdot \frac{r'^2}{r^2};$$

$$\therefore \; p = g\,h - \frac{r'^4}{r^4} \cdot g\,H + \Pi.$$

13.   A reflector is formed of an indefinite number of faces, which are tangents to a parabola, and rays diverge from a given point in the axis of the parabola; find the locus of the images produced by reflection.

Taking the radiating point O for the origin, let its distance AO from the vertex A of the parabola $= b$; then if $a =$ latus rectum, the equation of the parabola will be

$$y'^2 = a\,(x' + b). \qquad (1).$$

Let a small pencil of rays be incident at a point M in the parabola; draw the normal ML meeting the axis in L, and produce it to an indefinite distance MK in the opposite direction: draw $MP = MO$, making the $\angle\,PMK = \angle\,OML$; P is the place of the image, or a point in the curve required.

Whence, if x, y be the co-ordinates of P, we have, since $MP = MO$,

$$(y - y')^2 + (x - x')^2 = x'^2 + y'^2,$$
$$\text{or} \quad y^2 + x^2 = 2\,x\,x' + 2\,y\,y'. \qquad (2).$$

Also, since from the above construction, PO is parallel to ML; the $\angle\,AOP = \angle\,ALM$;

$$\therefore -\frac{y}{x} = \frac{d\,x'}{d\,y'} = \frac{2\,y'}{a}. \qquad (3).$$

Eliminating x' and y' from the equations (1), (2), (3), we arrive at the equation

$$y^2 = -\frac{2\,x^3 + 4\,b\,x^2}{2\,x + a},$$

which manifestly admits of only negative values of x; but if we reckon $+ x$ in the opposite direction, that is, change $+x$ into $- x$ the equation becomes

$$y^2 = \frac{4\,b\,x^2 - 2\,x^3}{2\,x - a},$$

which represents a curve of the conchoidal kind, belonging to Newton's 44th species of lines of the third order.

14. Three given quantities $(a + z)$, $(a + z) + h$, $(a + z) + h'$, approximations to the root a of an equation, being substituted for the unknown quantity, give results n, $n + \hat{c}$, $n + \hat{c}'$; shew that z will be very nearly found from the equation

$$z^2(h\,\hat{c}' - h'\,\hat{c}) + z(h^2\,\hat{c}' - h'^2\,\hat{c}) + n\,h\,h'(h' - h) = 0.$$

Let the equation be $f(x) = 0$; from which, by the conditions of the problem, we deduce the four following :

$$f(a) \quad = 0; \qquad f\left\{(a + z) + h\right\} = n + \hat{c};$$
$$f(a + z) = n; \qquad f\left\{(a + z) + h'\right\} = n + \hat{c}';$$

the three last of which, when expanded as far as the squares of the small quantities z, h, h', become

$$f(a) + f'(a) \cdot z + f''(a) \cdot \frac{z^2}{2} = n,$$

$$f(a + z) + f'(a + z) \cdot h + f''(a + z) \cdot \frac{h^2}{2} = n + \hat{c},$$

$$f(a + z) + f'(a + z) \cdot h' + f''(a + z) \cdot \frac{h'^2}{2} = n + \hat{c}';$$

which last equations, when their terms are expanded as far as to small quantities of the second order of approximation, and reduced by the three first, finally become

$$f'(a) \cdot z + f''(a) \cdot \frac{z^2}{2} = n, \tag{1}$$

$$f'(a) \cdot h + f''(a) \cdot z\,h + f''(a) \cdot \frac{h^2}{2} = \hat{c}, \tag{2}$$

$$f'(a) \cdot h' + f''(a) \cdot z\,h' + f''(a) \cdot \frac{h'^2}{2} = \hat{c}'. \tag{3}$$

Multiplying (3) by h, (2) by h', and subtracting, we get

$$f''(a) = 2\frac{(h\,\delta' - h'\,\delta)}{h\,h'\,(h'-h)};$$

which, being substituted in (2), gives

$$f'(a) = \frac{\delta}{h} - \frac{(h\,\delta' - h'\,\delta)\,(2z+h)}{h\,h'\,(h'-h)};$$

and, by substituting these values of $f'(a)$, $f''(a)$ in (1) we have

$$z^2\,(h\,\delta' - h'\,\delta) + z\,(h^2\,\delta' - h'^2\,\delta) + n\,h\,h'\,(h'-h) = 0.$$

Q. E. D.

15. If a plumb line be drawn from the vertical by a small quantity of matter, at a small elevation m above the Earth's surface, shew that the deviation will be a maximum, when $\sin\theta = \dfrac{m}{r\sqrt{2}}$ where θ is the angular distance at the Earth's centre of the plumb line and attracting point, and r the radius of the Earth supposed spherical.

Let a be the distance of the attracting point from the earth's centre, μ its mass, that of the earth being unity: then the distance of the attracting point from the pendulum

$$= \sqrt{(a^2 + r^2 - 2\,a\,r\,.\,\cos\theta)};$$

$\therefore$ its attractive force $= \dfrac{\mu}{a^2 + r^2 - 2\,a\,r\,.\,\cos\theta}.$

Now the deviation will evidently be a maximum when the resolved part of this force, in the direction of the horizon of the place at which the pendulum is situated, is a maximum. This resolved part of the attraction

$$= \frac{\mu\,r\,\sin\theta}{(a^2 + r^2 - 2\,a\,r\,.\,\cos\theta)^{\frac{3}{2}}};$$

which is to be a maximum;

$$\therefore \frac{\cos \theta}{(r^2 + a^2 - 2\,a\,r \cos \theta)^{\frac{3}{2}}} - \frac{3\,a\,r \sin^2 \theta}{(r^2 + a^2 - 2\,a\,r \cos \theta)^{\frac{5}{2}}} = 0;$$

$$\therefore \cos^2 \theta + \frac{r^2 + a^2}{2\,a\,r} \cdot \cos \theta = 3;$$

whence $\cos \theta = \sqrt{\left\{3 + \left(\frac{r^2 + a^2}{2\,a\,r}\right)^2\right\}} - \frac{r^2 + a^2}{2\,a\,r}.$

But $\dfrac{r^2 + a^2}{2\,a\,r} = 1 + \dfrac{r^2 + a^2}{2\,a\,r} - 1 = 1 - \dfrac{(a - r)^2}{2\,a\,r} = 1 - \dfrac{m^2}{2\,a\,r};$

$$\therefore \cos \theta = \sqrt{\left\{4 - \frac{m^2}{a\,r}\right\}} - \left(1 - \frac{m^2}{2\,a\,r}\right)$$

$$= 1 - \frac{m^2}{4\,a\,r}\text{ nearly};$$

rejecting all the powers of m above the square;

$$\therefore \sin \theta = \frac{m}{\sqrt{(2\,a\,r)}}.$$

$$= \frac{m}{r\sqrt{(2)}}\text{ nearly.}$$

Q. E. D.

16. A body drawn towards the origin of rectangular co-ordinates by a force $m\,r$, and from the plane $y\,z$ by a force $n\,x$, is projected perpendicularly to the plane of $x\,z$, from a point in it distant by D from each of the axes of x and z, with the velocity $\sqrt{(m)} \cdot D$; required the equations of the orbit, and the positions of the apses.

This problem has three cases according as m is greater, equal to, or less than n. First, let m be greater than n; then, since the resolved parts of the force $m\,r$ in the directions of x, y, z are respectively $-m\,x$, $-m\,y$, and $-m\,z$; the conditions of the problem give the equations

$$\frac{d^2 x}{d t^2} = - (m - n) \, x, \qquad (1)$$

$$\frac{d^2 y}{d t^2} = - m \, y, \qquad (2)$$

$$\frac{d^2 z}{d t^2} = - m \, z. \qquad (3)$$

Multiplying (1) by $2 \, dx$ and integrating, we get

$$\frac{d x^2}{d t^2} = C - (m - n) \, x^2;$$

but since the motion is at first wholly in the direction of y, which is when $x = D$; we have $0 = C - (m - n) \, D^2$;

$$\therefore \frac{d x^2}{d t^2} = (m - n) \, (D^2 - x^2);$$

$$\therefore t = \frac{1}{\sqrt{m - n}} \int \frac{d x}{\sqrt{D^2 - x^2}} + C,$$

$$= \frac{1}{\sqrt{m - n}} \cdot \sin^{-1} \frac{x}{D} + C,$$

$$= \frac{1}{\sqrt{m - n}} \left\{ \sin^{-1} \frac{x}{D} - \frac{\pi}{2} \right\}; \quad (4)$$

because at the beginning of the motion $x = D$.

Similarly, from equations (2) and (3) we get

$$t = \frac{1}{\sqrt{m}} \cdot \sin^{-1} \frac{y}{D}, \qquad (5)$$

$$t = \frac{1}{\sqrt{m}} \cdot \left\{ \sin^{-1} \frac{z}{D} - \frac{\pi}{2} \right\}. \qquad (6)$$

Eliminating t from (5) and (6) gives

$$\sin^{-1} \frac{y}{D} = \sin^{-1} \frac{z}{D} - \frac{\pi}{2},$$

or $\quad y^2 + z^2 = D^2,$

which shews that the body is always in the surface of a cylinder whose axis is that of x, and whose radius $= D$. The necessity of this conclusion is easily seen by considering that the force towards the axis is as the distance (as appears from equations (2) and (3)) and the velocity of projection is that which would retain it in a circle at a distance D from the axis, if no forces acted in the direction of x: but as the force in the direction of the axis of x cannot affect the motion towards or from that axis, the distance from it must remain invariable: that is, the body must move in the surface of a cylinder whose radius $= D$.

Also, by eliminating t from equations (4) and (5),

$$\frac{x}{D} = \cos\left\{\sqrt{\left(\frac{m-n}{m}\right)} \cdot \sin^{-1}\frac{y}{D}\right\},$$

which gives the projection of the curve on the plane of x, y: let θ be the angle made by the projection of r on the plane of $y\,z$, then since this projection $= D$; $y = D\sin\theta$;

$$\therefore \frac{x}{D} = \cos\left\{\sqrt{\left(\frac{m-n}{m}\right)} \cdot \theta\right\}. \qquad (6)$$

From this equation it appears that the greatest value of x, whether positive or negative, is D; the path of the body is, therefore a re-entering curve, which becomes an ellipse when $n = 0$, since in that case, the path, when the cylindrical surface is developed, becomes (by equation (6)) a figure of sines; which is a well known property of the ellipse; but this is also manifest from the consideration of centripetal forces of which the problem then becomes a common case.

To find the apses, (or points where the path is in a plane perpendicular to the axis of x,) we have only to make $\dfrac{d\,x}{d\,\theta} = 0$, which is satisfied by the following values of θ;

$$0,\ \pi\sqrt{\left(\frac{m-n}{m}\right)},\ 2\pi\sqrt{\left(\frac{m-n}{m}\right)}\ldots\ldots$$

corresponding to which the values of x are

$$D, \; - D, \; D, \; \ldots \ldots$$

which shews that there are only two apses.

When n is greater than m; we find, by a similar route to that pursued in the first case, that x becomes a logarithmic function of y or θ; so that the curve is then a spiral in the same cylindrical surface, and the number of apses is infinite.

When $n = m$, the curve is evidently a circle whose radius $= D$, in a plane parallel to that of $y\,z$ and at a distance from it $= D$.

17. If the particles of a fluid mass, revolving with a given angular velocity, be attracted to a fixed centre by a force, which is any function of the distance from the centre, the ellipticity, supposed small, will be half the ratio of the centrifugal force to the attraction at the equator.

If the fluid in a canal from the pole to the centre communicate with that in a canal from the equator to the centre, the pressures of each on a point at the centre, must, from the nature of a fluid, be equal. To determine these pressures, let a be the equatoreal and b the polar radius of the spheroid, ω its angular velocity; and let $\phi\,(r)$ be the attraction at a distance r from the centre.

Now, the pressure of the fluid at the centre in each canal is equal to the sum of all the forces that act on every particle of the fluid in each towards the centre; then, supposing the section of each canal $= 1$; the sum of all the forces in the polar canal

$$= \int \phi\,(r)\,.\,d\,r \begin{Bmatrix} r = 0 \\ r = b \end{Bmatrix} = \phi\,'(b);$$

taking $\phi\,'(r)$ to denote the integral of $\phi\,(r)$ from $r = 0$, to $r = r$.

But the sum of the forces in the equatoreal canal

$$= \int \phi(r)\, dr - \omega^2 \int r\, dr = \phi'(a) - \frac{\omega^2 a^2}{2};$$

$$\therefore \phi'(b) = \phi'(a) - \frac{\omega^2 a^2}{2},$$

$$\text{or } \phi'(a) - \phi'(b) = \frac{\omega^2 a^2}{2}.$$

But if ϵ denote the ellipticity, $b = \dfrac{a}{1+\epsilon} = a - a\,\epsilon$ nearly;

$$\therefore \phi'(b) = \phi'(a - a\,\epsilon) = \phi'(a) - \phi(a)\,.\,a\,\epsilon \text{ nearly}:$$

$$\therefore \phi'(a) - \phi'(b) = \phi(a)\,.\,a\,\epsilon = \frac{\omega^2 a^2}{2};$$

$$\therefore \epsilon = \tfrac{1}{2}\cdot\frac{\omega^2 a}{\phi(a)}$$

$$= \frac{1}{2}\cdot\frac{\text{centrifugal force}}{\text{att}^n\text{. at equator}}.$$

<div align="right">Q. E. D.</div>

18. Demonstrate the laws of reflexion and refraction on the undulatory theory of light.

See No. 24, of the Friday Evening Problems.

SATURDAY EVENING.

1. If S be the focus, and A the vertex of any conic section, and if LT the tangent at the extremity of the latus rectum L meet the axis in T; shew that $\dfrac{AS}{AT} =$ the eccentricity.

Let $r = SP$, $\theta = \angle ASP$; P being any point in the conic section; $2p$ = latus rectum; e = eccentricity.

Now $AT = ST - AS = SL \cdot \tan \angle SLT - AS$;

But, since $r = \dfrac{p}{1 + e \cos \theta}$; $SL = p$, $AS = \dfrac{p}{1 + e}$,

$\tan \angle SLT = \dfrac{r\, d\theta}{d r} = \dfrac{1 + e \cos \theta}{e \sin \theta} = \dfrac{1}{e}$; (because, at L, $\theta = 90°$.)

$$\therefore AT = p \left(\frac{1}{e} - \frac{1}{1 + e} \right) = \frac{p}{e (1 + e)};$$

$$\therefore \frac{AS}{AT} = \frac{p}{1 + e} \cdot \frac{e (1 + e)}{p} = e.$$

<div align="right">Q. E. D.</div>

2. From a station B at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit up a plane making 30° with the horizon, to another station C, the angle BCA is observed to be 135°. Find the height of the mountain in yards.

From A let fall a perpendicular AD meeting the horizontal line BD in D; from C let fall a perpendicular CE meeting BD in E; draw CF parallel to BD meeting AD in F.

In the right-angled triangle BEC we have BC = one mile, $\angle CBE = 30°$; $\therefore CE = \frac{1}{2}$ a mile.

In the triangle ACB we have given BC = one mile,

$$\angle ABC = \angle ABD - \angle CBE = 30°;$$
$$\angle CAB = \angle BCA - \angle ABC = 15°;$$
$$\therefore AC = BC \cdot \frac{\sin 30°}{\sin 15°} = 2^m \cdot \cos 15°.$$

In the right-angled triangle AFC, we have

$$AF = AC . \cos 15°$$
$$= 2^m . \cos^2 15 = \frac{2 + \sqrt{3}}{2} . \text{ miles};$$

$\therefore$ height required

$$= AF + FD = AF + CE = \frac{1}{2}(3 + \sqrt{3}) \text{ miles},$$
$$= 880 (3 + \sqrt{3}) \text{ yards},$$
$$= 4164 . 6 \text{ yards}.$$

3. A mortgage is taken on an estate worth N acres of it; land rises n per cent. in price, and in consequence the mortgage is only worth N_1 acres, and it is then paid off. During the continuance of high prices another mortgage is taken, which is worth N acres as before; prices return to their former level, and the mortgage is worth N_2 acres; shew that

$$N - N_1 : N_2 - N :: 1 : 1 + \frac{n}{100} .$$

Let $a = 1^{st}$ price of an acre in pounds; then

$$a + \frac{n \, a}{100} = 2^{nd} \text{ price };$$

$\therefore$ amount of the 1st mortgage $= N a = N_1 \left(a + \frac{n \, a}{100} \right)$

by the given condition;

$$\therefore N = N_1 + \frac{n \, a}{100} N_1, \text{ or } N - N_1 = \frac{n \, a}{100} N_1 .$$

Also, the amount of the 2^d mortgage $= N \left(a + \frac{n \, a}{100} \right)$

$$= N_2 \, a, \text{ by the given condition};$$

$$\therefore N_2 = N + \frac{n\,a}{100}\,N,\ \text{or}\ N_2 - N = \frac{n\,a}{100}\,N = \frac{n\,a}{100}\Big(1 + \frac{n}{100}\Big)N_1;$$

$$\therefore N - N_1 : N_2 - N :: \frac{n\,a}{100}\,N_1 : \frac{n\,a}{100}\Big(1 + \frac{n}{100}\Big)N_1$$

$$:: 1 : 1 + \frac{n}{100}$$

<div align="right">Q. E. D.</div>

From this result it appears that the advantage to the mort-gager from a rise in the price of land is less than the dis-advantage to the mortgagee from a fall in price, in the ratio of 1 to $1 + \frac{n}{100}$.

4. The distance of a point P from the circumference of a circle : its distance from a fixed diameter $AB = n : 1$. Prove that the locus of P is a conic section.

Let C be the centre of the circle; PQ the distance of P from the circumference; let fall the perpendiculars PM, QN on the diameter AB.

Let $AC = a$, $CM = x$, $CN = x'$;
$$PM = y,\ QN = y'.$$

Then $PQ^2 : PM^2 :: n^2 : 1$; or $PQ^2 = n^2 \cdot PM^2$.

But $PQ^2 = (x' - x)^2 + (y' - y)^2$;
$$\therefore (x' - x)^2 + (y' - y)^2 = n^2 y^2,$$
or $\quad x'^2 + y'^2 + x^2 + y^2 - 2(x\,x' + y\,y') = n^2 y^2,$
or $\qquad a^2 + x^2 + y^2 - 2(x\,x' + y\,y') = n^2 y^2.$

Also, from the similar triangles CNQ, CMP;

$$\frac{CQ}{CN} = \frac{CP}{CM},\ \text{or}\ \frac{a}{x'} = \frac{\sqrt{(x^2 + y^2)}}{x};\ \ \therefore x' = \frac{a\,x}{\sqrt{(x^2 + y^2)}}.$$

This value of x' being substituted in the last equation, and also in $y'^2 + x'^2 = a^2$; and y' being eliminated from both; we arrive, after the requisite reductions, at the equation

$$(n^2 - 1)\,y^2 - 2\,n\,a\,y - x^2 + a^2 = 0,$$

which is that of a conic section, one of whose axes is the axis of y.

When $n > 1$, the equation represents an hyperbola,

when $n < 1$, an ellipse, and when $n = 1$, a parabola.

5. An indefinite area is to be divided into similar and equal *regular* figures. Shew by what figures this can be done. Also, if three equal areas be divided into the same number of equal *regular* figures which are respectively triangular, square, and hexagonal, shew that the sum of the lengths of the dividing lines in the cases of triangular, square, and hexagonal divisions, are to one another as $\sqrt[4]{27}$, $\sqrt[4]{16}$, $\sqrt[4]{12}$; the whole area to be divided being very great in comparison of one of the divisions.

If n be the number of sides of any regular polygon, the angle subtended by each side at the centre of the circumscribing circle is $\dfrac{2\,\pi}{n}$; and therefore the angle at each summit of the figure

$$= \pi - \frac{2\,\pi}{n} \text{ or } \frac{n-2}{n}\,\pi\,.$$

That an indefinitely extended area may be divided into similar and equal regular figures, the angle at every summit of each figure must evidently be an exact measure of four right angles:

$$\therefore \text{ if } m \cdot \frac{n-2}{n}\,\pi = 2\,\pi, \text{ or } m \cdot \frac{n-2}{n} = 2,$$

D

m or $\dfrac{2\,n}{n-2}$ must be a whole number; $\therefore$ $m-2$ or $\dfrac{4}{n-2}$ must be a whole number. But the only values of n which can make $\dfrac{4}{n-2}$ a whole number are 3, 4, and 6; therefore equilateral triangles, squares, and hexagons are the only figures possessing the required property.

The indefinite and equal areas being divided into the same number of equal and regular figures, the areas of each of these figures must be equal in each case; and the sum of the dividing lines in each case must be as the length of a side of one of the regular figures multiplied into the number of its sides. Let s_3, s_4, s_6 denote respectively the lengths of the sides of an equilateral triangle, square, and hexagon of equal areas: S_3, S_4, S_6 the sum of the dividing lines in the respective cases; then since the area of a polygon of n sides, each of which $= a$,

$$\text{is } \frac{n\,a^2}{4}\cot\frac{\pi}{n};$$

we have, by putting for n, 3, 4, 6, and for a, S_3, S_4, S_6 successively,

$$\frac{3\,s_3^2}{4}\cdot\cot 60^\circ = \frac{4\cdot s_4^2}{4}\cot 45^\circ = \frac{6\cdot s_6^2}{4}\cot 30^\circ;$$

$$\text{or } \sqrt{3}\cdot s_3^2 = 4\cdot s_4^2 = 6\sqrt{3}\cdot s_6^2;$$

$$\therefore \frac{s_3}{s_4} = \frac{2}{\sqrt[4]{3}}; \ \therefore \frac{S_3}{S_4} = \frac{3\cdot 2}{4\sqrt[4]{3}} = \frac{\sqrt[4]{3^3}}{\sqrt[4]{2^4}} = \frac{\sqrt[4]{27}}{\sqrt[4]{16}}.$$

Again,

$$\frac{s_3}{s_6} = \sqrt{6}; \ \therefore \frac{S_3}{S_6} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt[4]{3}\cdot\sqrt[4]{9}}{\sqrt[4]{3}\cdot\sqrt[4]{4}} = \frac{\sqrt[4]{27}}{\sqrt[4]{12}}:$$

$$\therefore S_3 : S_4 : S_6 :: \sqrt[4]{27} : \sqrt[4]{16} : \sqrt[4]{12}.$$

Q. E. D.

From this result we observe that when the area is divided into hexagons, the sum of the dividing lines is a minimum. Thus a piece of fine and regular net-work may be made with the least quantity of material when the meshes are of a hexagonal form.

If instead of the indefinite area, we conceive an indefinite solid to be divided into regular solid prisms whose axes are all parallel to one another, the same reason will evidently shew that the dividing surfaces are a minimum when the prisms are hexagonal. This is precisely the case in the structure of the honeycomb, in which the cells of wax are six-sided prisms; and, therefore, are so arranged as to contain the greatest quantity of honey in the least quantity of wax.

6. In a given equilateral parallelogram inscribe an ellipse of given eccentricity.

Let a' be the side of the equilateral parallelogram, r one of its angles; a, b the semi-axes of the ellipse, and e its eccentricity.

Since all parallelograms circumscribing an ellipse are equal;

$$a'^2 \sin \gamma = 4\, a\, b = 4\, a^2 \sqrt{(1 - e^2)};$$

$$\therefore a^2 = \frac{a'^2 \sin \gamma}{4 \sqrt{(1 - e^2)}}; \text{ and } b^2 = a^2 (1 - e^2) = \frac{a'^2}{4} \sin \gamma \sqrt{(1 - e^2)}$$

therefore the magnitudes of the axes of the ellipse are known; and since one of them always lies in one of the diagonals of the parallelogram; their position is also known.

7. If A and a be the areas of horizontal sections of a waterfall, at heights H and h above the horizon, find the height of the fall, the (velocity)² of the water at any point being as the height from which it has fallen.

The horizontal sections being inversely as the velocities (Moseley's Hydrostatics, Art. 136.), if x denote the height of the fall,

$$\frac{A^2}{a^2} = \frac{x - h}{x - H}; \quad \therefore x = \frac{A^2 H - a^2 h}{A^2 - a^2}.$$

8. In a given latitude a vertical rod is placed at a given distance from an East and West wall so as to cast a portion of its shadow upon it, find the equation to the extremity of the shadow traced upon the wall on a given day. Shew what the equation becomes when the Sun is in the Equator and the latitude of the place 45°.

If a, δ, z, λ denote respectively the sun's azimuth, declination, zenith-distance, and the latitude of the place, we have

$$\cos a = \frac{\sin \delta - \cos z \cdot \sin \lambda}{\sin z \cdot \cos \lambda} = \frac{\sin \delta}{\cos \lambda \cdot \sin z} - \frac{\tan \lambda}{\tan z}.$$

Let the origin of co-ordinates be the point where a perpendicular $= b$, from the top of the rod, meets the plane of the wall; and let the positive direction of y be reckoned vertically downwards; that of x being horizontal and east-ward.

We then have $\tan a = \dfrac{x}{b}$, $\tan z = \dfrac{b}{y}$;

$$\therefore \cos a = \frac{b}{\sqrt{(b^2 + x^2)}};$$

$$\text{and } \sin z = \frac{b}{\sqrt{(b^2 + y^2)}};$$

therefore by substituting these values in the above equation;

$$\frac{b}{\sqrt{(b^2 + x^2)}} = \frac{\sin \delta}{b \cdot \cos \lambda} \cdot \sqrt{(b^2 + y^2)} - \frac{y}{b} \cdot \tan \lambda,$$

which is the equation to the path of the extremity of the shadow.

When the Sun is in the Equator, and the latitude $45°$; $\delta = 0$, and $\lambda = 45°$; therefore the equation becomes

$$y^2 = \frac{b^4}{b^2 + x^2},$$

which represents a curve of the conchoidal kind, to which the axis of x is an asymptote, and the infinite branches equal and similar on each side of the plane of the meridian, as might have been expected.

9. Find the time of emptying a given prismatic vessel filled with water, by a cycloidal syphon of small bore placed with its base horizontal; the vertex of the syphon resting on the edge of the vessel.

Let BAC be a cycloid whose vertex is A, base BC, and vertical axis AD. From any point P in AC let the horizontal ordinate PN be drawn meeting AD in N.

Let $AD = h$, $AN = x$, $AP = s$; s being reckoned positively from A towards C, and negatively from A towards B.

Let NP be the position of the surface of the water in the prismatic vessel at the end of any time t.

Now since the pressure at the atmosphere at the orifice B and at the surface NP retains the water in the tube BAP in one mass, so that every part of it flows with the same velocity at the same time; the motion of the mass BAP is at any given instant the same as that of a solid; and may therefore be determined by the principle of D'Alembert. To apply this, let k be the area of any transverse section of the tube; then the mass of any element ds (its density being unity) will $= k\,ds$, and if α be the angle of its inclination to the horizon, its impressed moving force will be $g \sin \alpha \,.\, k\,ds$; also if f be its effective accelerating force, its effective moving force will be $f \,.\, k\,ds$:

therefore as all such forces are in equilibrium and all act in the same line,

$$\int g \sin a \cdot k \, ds = \int f \cdot k \, a \, s.$$

But $\sin a = \dfrac{d\,x}{d\,s}$; and f being, at any given time, the same for every element from B to P, may be placed without the sign of integration:

$$\therefore g \int \frac{d\,x}{d\,s} \cdot ds = \int f \, ds,$$

the limits of each integral being from $s = -AB \, (= -2\,h)$ to $s = AP$.

Also, in the cycloid $\dfrac{d\,x}{d\,s} = \dfrac{s}{2\,h}$; $\therefore$ the above equation finally becomes

$$\frac{g}{4\,h}\,(s^2 - 4\,h^2) = f\,(s + 2\,h); \; \therefore f = \frac{g}{4\,h}\,(s - 2\,h).$$

Let now v be the velocity in the syphon, V that of the descending surface in the prismatic vessel, and K the area of that surface: then, since $v \, dv = -f \, ds$, we have

$$v^2 = \frac{g}{4\,h}\,(4\,h\,s - s^2) + C:$$

but at the beginning of the motion $s = 0$, $\therefore C = 0$.

Again $V = \dfrac{k}{K} \cdot v$;

$$\therefore t = \frac{2\,K}{k}\,\sqrt{\left(\frac{h}{g}\right)} \cdot \text{ver} \sin^{-1} \frac{s}{2\,h},$$

which, for the whole time of emptying, at the end of which $s = 2\,h$, becomes

$$t = \frac{K}{k} \cdot \pi\,\sqrt{\left(\frac{h}{g}\right)}.$$

The equation $g \int \dfrac{d\,x}{d\,s} \cdot ds = \int f \, ds$, determines the accele-

rating force in any kind of syphon of small bore, in which the origin of co-ordinates is at the highest point. In the above example we have considered s the independent variable; but in other cases it will mostly be convenient to consider $\frac{d\,x}{d\,s}$ and s functions of x, in which case we should have

$$g\,x = f\,s + \text{constant}$$

which taken between the same limits as before gives

$$g\,(A\,D - A\,N) = f\,(A\,B + A\,P),$$

from which we gather that, the mass in motion being as $AB + AB$, the moving force is as $AD - AN$, or, in all cases, as the difference of level between the longer and shorter legs.

10. A small pencil of diverging rays is incident obliquely on a reflecting surface at its center, and the section of the surface in the primary plane is a circle of given curvature; find the curvature of the section in the secondary plane, that the foci of rays reflected in the two planes may coincide.

If we conceive two spheres, whose radii are r_1 and r_2, to oscillate, at the point of reflexion, in the directions of the primary and secondary planes; ϕ being the angle of inclination of the incident pencil with the normal, u its length, v_1 that of the reflected pencil in the primary plane, for the first sphere; v_2 the length of the reflected pencil in the secondary plane, for the second sphere: we then have, by considering each sphere as a reflector separately,

$$\left(\frac{1}{u} + \frac{1}{v_1} \right) \cos \phi = \frac{2}{r_1}; \quad \frac{1}{u} + \frac{1}{v_2} = \frac{2 \cos \phi}{r_2}.$$

(*Coddington on Reflexion and Refraction*).

But as the curvature of either section cannot affect the focal distance in the plane of the other; these two equations will be true simultaneously, when we replace the two spheres by the surface above described : but, since by hypothesis, the foci in the primary and secondary planes coincide, we have $v_1 = v_2$; and therefore the two equations combined give

$$r_2 = r_1 \cos^2 \phi.$$

11. If a uniform chain be suspended from two piers, the points of suspension being in the same horizontal line, shew that when the chain is nearly horizontal, the tension is nearly equal to the weight of a length $\dfrac{s^2}{4(s-b)}$ of the same chain, where $s = $ length of the chain, and $b = $ the distance between the points of suspension. Shew also that such a length may be given to the chain as to render the tension at either pier a minimum, and investigate an equation for determining the minimum tension.

If h be the sagitta of the arc, and c the horizontal tension ; the common property of the catenary gives $\left(\dfrac{s}{2}\right)^2 = 2ch + h^2$, and the tension at the points of suspension $= c + h = c$ nearly, h being very small in comparison of c.

The tension is, therefore, represented in this case by

$$\frac{s^2 - 4h^2}{8h} = \frac{s^2}{8h} \text{ nearly.}$$

But as the arc $\frac{s}{2}$ nearly coincides with its chord, on account of the smallness of h; we have $4\,h^2 = s^2 - b^2$.

Now $(s - b)^2 = s^2 + b^2 - 2\,s\,b$, which becomes the more nearly equal to $s^2 - b^2$ as s becomes more nearly equal to b; because $- 2\,s\,b$ then approximates to $- 2\,b^2$. The above expression for the tension thus becomes reduced to $\dfrac{s^2}{4\,(s - b)}$.

That there is a certain length of chain corresponding to a minimum tension, appears from the following considerations:

If the chain be perfectly straight and horizontal, the tension is infinite, as appears by making $s = b$, in its expression found above: the least quantity of chain, therefore, gives the greatest tension. But if we increase the length to a very great extent, it will tend to become vertical at the points of suspension, and the tension will in consequence tend to become equal to half the weight of the chain, and therefore to become again indefinitely great. Since, then, both the least and greatest quantities of chain give the greatest possible tensions; it necessarily follows that there must exist some intermediate length of chain which shall give a minimum tension.

To get an equation for determining this, we have the well-known expression $s = c\,(e^{\frac{y}{c}} - e^{-\frac{y}{c}})$, (Whewell's Mechanics,) and the tension is expressed by $c + x$ or $\sqrt{c^2 + \dfrac{s^2}{4}}$ which is to be a minimum; and therefore $c^2 + \dfrac{s^2}{4}$ is to be a minimum;

$$\text{or } c^2 + \frac{c^2}{4}\,(e^{\frac{b}{c}} + e^{-\frac{b}{c}} - 2).$$

This last equation being differentiated with respect to c, and the co-efficient of $d\,c$ made $= 0$, gives an equation for determining c; which being found and substituted in the equation

$$s = c \, (e^{\frac{b}{qc}} - e^{-\frac{b}{qc}}),$$

gives the required length s. But the resolution of the equation for determining c is not practicable.

12. Find the quantities of matter of planets which have satellites ; and these being given, shew how to determine the quantities of matter of those which have not satellites.

The *mass* of one body is said to be as much greater than that of another, as the moving force of gravity in the one, is observed to exceed that in the other. Now, the mass is defined to be the measure of the actual quantity of matter, of which the body is composed; but of this magnitude we have no means of judging but by its mechanical effect; and in this we manifestly assume, that the mechanical influence which produces this effect must be exerted equally in every equal subdivision of the mass. Thus, when two heavy bodies sustain each other by means of a machine, so constructed, that forces applied in the places of the bodies must have a certain ratio to one another, we say, that the masses of the bodies are to one another in that same ratio; thereby assuming, that the moving forces of gravity in different bodies are always to one another in the ratio of their quantities of matter: how far this is confirmed by experience we shall presently show.

Another mechanical property, by which we estimate the quantity of matter, is *inertia*. To illustrate this, suppose the two bodies above mentioned to be set in motion on a perfectly smooth horizontal plane, by the impulsive force of the same spring acting on each body for an equal time: then, knowing the the relation between the accelerating force and velocity, in motion thus produced, we can compare the accelerating forces of the spring on each body, and thus compute the quantity of

inertia in each, which is found to exist in precisely the same ratio as the moving forces of gravity. Thus, two properties of matter which, abstractedly considered, do not appear to be *essentially* co-existent, are nevertheless found to exist together in the same ratio in every case in which it can be observed.

If this were not the case, the same moving force of gravity could no longer produce the same acceleration in different bodies. To illustrate this, let us suppose the equal and uniform force of gravity, near the earth's surface, to be applied severally and at the same time to every one of the equal particles of a mass of matter: it is clear that the mass must at the end of any time have acquired the same velocity, and described the same space as each of its particles would have done had they been separate or without cohesion: in other words, the velocity acquired and the space described must be, in this case, entirely independent of the quantity of matter; and therefore, whatever be the moving force of gravity, its accelerating force on the body must be invariable. If, on the contrary, we conceive to be added to the same mass, and mixed up with it, another equal, and endued with the same inertia, but not acted on by the force of gravity; the velocity acquired, and the space described, at the end of any time, will be only half of those in the first case; because the same influence that was at first exerted on only *one* particle, is now divided between *two*, and, therefore, by the experiment with the spring above mentioned, can only produce half the acceleration, although the moving force is in both cases the same. We are thus led to the conclusion, that if uniform gravity acted unequally on any equal masses of matter, a difference of acceleration must be the consequence: accordingly Newton having made experiments on a great variety of bodies, and found an equal acceleration in every case, concluded that all terrestrial matter was equally heavy.

It is scarcely necessary to remark, that this conclusion ap-

plies only to bodies equally distant from the earth's centre, or at distances whose differences are so small as to be inconsiderable with respect to the distances themselves; the law of the diminution of gravity being, that the intensity or accelerating force of a given mass upon another is inversely as the square of their mutual distance. To shew, therefore, that the inequality of the Earth's accelerative force, on any portion of the Moon's mass, arises only from the difference of distance from the Earth's centre, it is only necessary to shew, that the spaces descended through in one second by a body at the distance of the Moon from the Earth, and another near the Earth's surface are in the inverse ratio of the squares of the distances from the Earth's centre. But these two data are furnished by observing, from the dimensions of the Moon's orbit and her periodic time, the space through which a portion of her orbit, described in one second, is deflected from a tangent at that point at which the deflection is supposed to begin. It is thus ascertained that the Moon's mass cannot be made up partly of inert gravitating matter, and partly of inert matter which does not gravitate, or in other words, that the moving force of the whole Earth on any portion of the Moon's mass is as that portion.

From similar considerations, it is inferred that the intensity of gravity throughout the solar system follows the same law, and that an equal portion of the mass of any of the bodies in it would produce an equal accelerative effect upon a given mass at a given distance : the *absolutute intensity* therefore of the force of any attracting body depends only on the number of particles of matter it contains. It hence follows that the moving forces of any two planets on each other are equal; for if m, m' denote their masses and r their mutual distance, the moving force of m on $m' = \dfrac{m\,m'}{r^2}$, which also expresses the moving force of m' on m.

The accelerating force of m on m' is to that of m' on m as $\frac{m}{r^2}$ to $\frac{m'}{r^2}$, or as m to m'; and therefore the relative accelerating force of their mutual approach is as $\frac{m + m'}{r^2}$; which also expresses the centripetal force of one of the bodies towards the other, about which it is supposed to revolve as a fixed centre.

It is well known, that if a body move in an ellipse about a fixed centre, towards which it is impelled by a force varying inversely as the square of the distance, the periodic time of one revolution is expressed by $\frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$; where a is the mean distance of the revolving body, and μ the absolute force $= m + m'$, if m and m' denote the masses of the fixed and revolving bodies. To apply this to determine the ratio of the masses of the sun and a planet accompanied by a satellite, let m, m' be the masses of the planet and satellite; that of the sun being unity; let p, p' be the periodic times of the planet and satellite; r, r' the mean distances of the planet from the sun, and of the satellite from the planet : then, by the above formula,

$$\frac{p^2}{p'^2} = \frac{r^3}{r'^3} \cdot \frac{m + m'}{1 + m} = \frac{r^3}{r'^3} \cdot m$$

nearly; since m is very small in comparison of 1, and m' in comparison of m. Thus $m = \frac{r'^3}{r^2} \cdot \frac{p^2}{p'^2}$ nearly : and in this manner we might compare the masses of the Sun, Earth, Jupiter, Saturn, and Uranus; and the mass of the Earth, being determined, by an independent method, according to some substance of known density taken as a standard, (water for instance) we are able to assign the masses of these bodies in terms of a given quantity of this substance. This method, for a first approximation, is practicable enough; but in taking the actual case of any one of these bodies we find it attended by peculiarities which

vitiate the result, and which, therefore, require a modification of the process adapted to that case. To enumerate these different peculiarities, with the several methods of encountering them, would lead us very far beyond the limits of this article, the object of which is, merely to give a brief outline of the relative *possibilities* of determining the mechanical affections of the several bodies which compose our system; referring the reader, for the details, to the treatises on Physical Astronomy now in use; as well as to offer such general information as may give correct notions of the data with which they set out.

Among the peculiarities affecting the results afforded by this method, we may mention that in Jupiter and Saturn the oblateness of their figures causes a deviation, in the law of force towards their centres, from that which would hold if they were spherical: this deviation is probably much more augmented in the latter body by the attraction of its ring.

The masses of those planets which have not satellites must be determined from their mechanical action on each other in the following manner:

Let r, θ be the radius vector and longitude of a planet at any assigned time, computed on the supposition of its moving in an ellipse; and therefore not subject to any disturbing force; then, from theory, we can compute δr, $\delta \theta$, the variations or perturbations of these elements from the action of a body whose mass $= 1$, so that $m \delta r$, $m \delta \theta$ may represent what these effects become from a mass m.

Similarly, we can calculate $\delta' r$, $\delta' \theta$, $\delta'' r$, $\delta'' \theta$, &c. the perturbations resulting from other masses (each equal to unity) at other positions relatively to the disturbed body; so that $m' \delta' r$, $m' \delta' \theta$, $m'' \delta'' r$, &c. may represent what these effects become from masses m', m'', &c.

If, then, L, R be taken to represent the true values of the longitude and radius vector as deduced from observation, we have

$$\theta + m \cdot \partial \theta + m' \cdot \breve{\partial} \theta + m'' \cdot \breve{\partial}'' \theta + \ldots = L,$$

$$r + m \cdot \partial r + m' \cdot \breve{\partial}' r + m'' \cdot \breve{\partial}'' r + \ldots = R,$$

in which m, m', m'', &c. are the only unknown quantities. Now if we take either of these equations, (the first for instance) and compute several values of $\theta, \partial \theta$, &c. and obtaining, by observations, as many values of L; we may obtain as many equations for determining m, m', m'', as we please, remembering that such of the values of m, m', &c. as belong to planets having satellites may be regarded as known.

If all the data in this problem were perfectly accurate we might only deduce as many equations as we had unknown quantities to determine, and by increasing the number we should only get a repetition of the same results which would only be useful as far as they went to confirm one another : but this is far from being the case; the observations are in the first place imperfect, as well as the requisite data for reducing them, and thus the several equations, however numerous, would each give a somewhat different result. We may, however, advantageously employ a great number of equations, by combining them in such a manner that each may have its influence on the result, and the different errors be so blended as nearly to destroy one another.

This problem, and indeed every other in Physical Astronomy, practically consists of a set of successive approximations, each step of which is conducted by successive generations of Astronomers ; the results of each serving to correct the data of the next ; and though the whole rests upon one single and simple hypothesis, confirmed by successive observations into a *law* of nature, yet so numerous are the consequences of this law and so intricate in their connection that nothing short of a length of ages is requisite to develope them by observations, the ana-

lytical difficulties of the problem being too great to enable us to anticipate them by theory alone.

With regard to determining the masses of comets we may add, that hitherto little more has been done than to infer, from their insensible action on the planets, and the great derangement of their motions from the actions of the latter, that their masses must be extremely small compared with that of any one of the planets; so much so as to make it not improbable that their densities may be no greater than that of a fog or vapour.

It has been suggested that from this circumstance, we may expect to detect the existence of a resisting ether which might sensibly affect the motions of such a body; and calculations to this effect have already been made, but with what success remains yet to be decided.

13. The ages of a man and his wife are respectively $85 - m$ and $86 - n$, find the present worth of an annuity A to be paid to the wife after the death of her husband, supposing one male out of every 85 and and one female out of every 86 to die annually; and n greater than m.

It has been shown by Demoivre, Waring, and Emerson, that the present value of an annuity in such a case is equal to that of the annuity for the longer of the two lives diminished by that of the joint annuity for both lives together.

Let r be the amount of £1. for one year; then the probability that the wife will be alive for 1, 2, 3, &c. years will be $\dfrac{n-1}{n}$, $\dfrac{n-2}{n}$, $\dfrac{n-3}{n}$, &c.; and the present values of £1. for these years $\dfrac{n-1}{n\,r}$, $\dfrac{n-2}{n\,r^2}$, $\dfrac{n-3}{n\,r^3}$, &c.;

∴ the present value of A for the life of the Wife

$$= A \left\{ \frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} + \dots \text{ to } n \text{ terms} \right\}.$$

Again, the probabilities that the husband will be alive for 1, 2, 3, &c. years being $\frac{m-1}{m}$, $\frac{m-2}{m}$, $\frac{m-3}{m}$, &c.; the probabilities of their being *both* alive at the end of these years are $\frac{m-1}{m} \cdot \frac{n-1}{n}$, $\frac{m-2}{m} \cdot \frac{n-2}{n}$, &c.;

∴ the value of A for both lives

$$= A \left\{ \frac{m-1}{m} \cdot \frac{n-1}{nr} + \frac{m-2}{m} \cdot \frac{n-2}{nr^2} + \dots \text{ to } m \text{ terms} \right\};$$

∴ the present value required

$$= A \left\{ \frac{n-1}{nr} + \frac{n-2}{nr^2} + \dots \text{ to } n \text{ terms} \right\}$$

$$- A \left\{ \frac{m-1}{m} \cdot \frac{n-1}{nr} + \frac{m-2}{m} \cdot \frac{n-2}{nr^2} + \dots \text{ to } m \text{ terms} \right\}.$$

14. If AP be any curve referred to a pole S, and if u be the solid generated by the revolution of the area ASP about AS, $SP = r$, and the angle $ASP = \theta$,

shew that $\dfrac{du}{d\theta} = \dfrac{2}{3}\pi r^3 \cdot \sin\theta.$

Taking a point Q in the curve near to P, join SQ, and from G the centre of gravity of the area PSQ,* let fall the perpendicular GM upon AS.

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* It may be objected that we are here making use of a physical property to prove one which is purely geometrical; but it must be kept in mind that the point called the centre of gravity of a *physical* area is one whose position has a certain *geometrical* relation to the figure and dimensions of the *geometrical* area corresponding to it; and which exists independently of any

Let the $\angle\ MSG = \phi$, $\angle\ PSQ = \delta\theta$, and the solid generated by the area $PSQ = \delta u$.

Now the theorem of Guldinus gives

$$\delta u = \text{area } PSQ \times 2\,\pi\,GM$$
$$= \text{area } PSQ \times 2\,\pi \sin\phi\,.\,SG.$$

But when $\dfrac{\delta u}{\delta\theta}$ becomes $\dfrac{d\,u}{d\,\theta}$, we have

$$\text{area } PSQ = \frac{1}{2}\,.\,r^2\,d\,\theta,$$

$$SG = \frac{2}{3}\,.\,r,$$

$$\phi = \theta\,;$$

$$\therefore \frac{d\,u}{d\,\theta} = \frac{r^2}{2}\,.\,2\,\pi\sin\theta\,.\,\frac{2}{3}\,r = \frac{2}{3}\,\pi\,r^3\,.\,\sin\theta.$$

Q. E. D.

15. If two equal bodies which attract each other with forces varying as $\dfrac{1}{(\text{dist.})^2}$ are constrained to move in two straight lines at right angles to one another, shew that they will arrive together at the point of intersection of the lines, from whatever points their motions commence. And, having given their distance at the beginning of the motion, find the time to the point of intersection.

Let x be the distance of the body P from the line in which the body Q moves, y the distance of Q from the line in which P moves, and r the distance of P from Q, at the end of the time t.

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physical properties whatever. The term "centre of gravity" is here used only as an abbreviation of the definition of a geometrical point, whose position is determined by the same artificial methods as that of the point so called in a material area.

Let $\frac{m}{2}$ be the mass of each body; then the accelerating force of Q on $P = \frac{m}{r^2}$, and this resolved in the direction of its motion is $-\frac{m}{r^2} \cdot \frac{x}{r}$;

$$\therefore \frac{d^2 x}{d t^2} = -\frac{m x}{r^3};$$

similarly $\quad \dfrac{d^2 y}{d t^2} = -\dfrac{m y}{r^3}.$

Multipl ing the first of these equations by y, the second by x, and subtracting, we have $\dfrac{y\, d^2 x - x\, d^2 y}{d t^2} = 0$;

whence, by integration, $y \dfrac{d x}{d t} - x \dfrac{d y}{d t} = c.$

But $\dfrac{d x}{d t}$, $\dfrac{d y}{d t}$ are the velocities of P and Q at any time t, and as these velocities begin together, we must have $c = 0$;

$\therefore y\, d x - x\, d y = 0$, or by integration, $y = k x.$

Now the arbitrary constant k expresses the tangent of the inclination of the line PQ or r with the axis of x; and as this is constant, the line PQ must always move parallel to itself and therefore the points P, Q must necessarily come to the origin at the same instant.

To find the time of coming to the origin, let us resume the equations

$$\frac{d^2 x}{d t^2} = -\frac{m x}{r^3}, \; \frac{d^2 y}{d t^2} = -\frac{m y}{r^3};$$

which give, after multiplying the first by $2\, d x$, the second by $2\, d y$, and adding,

$$\frac{2\, d x\, d^2 x + 2\, d y\, d^2 y}{d t^2} = -2 m \frac{(x\, d x + y\, d y)}{r^3}$$

$$\text{or} \quad d \cdot \frac{d\,x^2 + d\,y^2}{d\,t^2} = -\,2\,m\,\frac{d\,r}{r^2},$$

$$\text{or} \quad \frac{d\,x^2 + d\,y^2}{d\,t^2} = C + 2\,\frac{m}{r}.$$

But $r = x\,\sqrt{(1 + k^2)}$, $\therefore d\,r^2 = d\,x^2\,(1 + k^2)$;

$$\therefore \frac{d\,x^2 + d\,y^2}{d\,t^2} = \frac{d\,x^2\,(1 + k^2)}{d\,t^2} = \frac{d\,r^2}{d\,t^2};$$

$$\therefore \frac{d\,r^2}{d\,t^2} = C + \frac{2\,m}{r}.$$

But $\dfrac{d\,r}{d\,t}$ is the velocity of diminution of the distance r; then supposing this distance $= a$ at the beginning of the motion, we have

$$0 = C + \frac{2\,m}{a};$$

$$\therefore \frac{d\,r^2}{d\,t^2} = 2\,m\left(\frac{1}{r} - \frac{1}{a}\right),$$

$$\therefore t = -\,\frac{1}{\sqrt{(2\,m)}} \cdot \int \frac{d\,r}{\sqrt{\left(\dfrac{1}{r} - \dfrac{1}{a}\right)}} \begin{Bmatrix} r = a \\ r = 0 \end{Bmatrix}$$

$$= \frac{\pi}{2} \cdot \frac{a^{\frac{3}{2}}}{\sqrt{(2\,m)}}.$$

Thus it appears that the time is the same as that in which one body falls freely to the other as to a fixed centre, from the same distance.

16. Shew that every differential equation of the nth order has n first integrals.*

Integrate

$$\frac{d\,x}{x^2\,\sqrt{2\,a\,x - x^2}}, \quad \frac{d\,\theta}{1 - e^2 \cdot \cos^2 \theta}, \quad x\,y^2\,d\,y + y^3\,d\,x = a^3\,d\,x,$$

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* See Lagrange, Calcul des Fonctions, p. 151.

$$\left.\begin{array}{l} \dfrac{d^2 x}{d t^2} = a - b \dfrac{d x^2}{d t^2} \\[2mm] \dfrac{d^2 y}{d t^2} = b \left(c - \dfrac{d y}{d t} \right)^2 \end{array}\right\} \text{ (eliminate } t \text{, supposing } x, y, t,$$
$$\dfrac{d x}{d t}, \dfrac{d y}{d t} \text{ to vanish together.)}$$

$$\left(\frac{d z}{d x} \right)^2 + \left(\frac{d z}{d y} \right)^2 = a^2.$$

Before integrating $d u = \dfrac{d x}{x^2 \sqrt{a x - x^2}}$ we shall first give a general and practical rule for the integration of binomial differentials of the form $d u = x^{m-1} d x (a + b x^n)^p$.

Assume in all cases $P = x^m (a + b x)^{p+1}$, *except*

(1) When you want to reduce m down, for m write $m - n$;

(2) When you want to reduce p down, for $p + 1$ write p.

Differentiate the expression for P, and then rationalize the irrational numerators in its differential, and lastly integrate the expression so reduced, and an equation results for determining u in terms of an integral of an inferior order of reduction, or in which m or p is increased or diminished as required. This reduction to be repeated as often as necessary.

Observe, that by every reduction of the index of x, this index is increased or diminished by n; and, by every reduction of p, p is increased or diminished by 1.

To apply this to the case proposed, we must first reduce it to the form $x^{m-1} d x (a + b x^m)^p$; for which let $x = \dfrac{1}{v}$;

$$\therefore d u = - \frac{v \, d v}{\sqrt{(2 a v - 1)}} :$$

And, as this comes under the exception (1), we have

$$P = v \sqrt{(2 a v - 1)};$$

$$\therefore \frac{d P}{d v} = \sqrt{(2 a v - 1)} + \frac{a v}{\sqrt{(2 a v - 1)}},$$

$$= \frac{3\,a\,v}{\sqrt{(2\,a\,v-1)}} - \frac{1}{\sqrt{(2\,a\,v-1)}},$$

by rationalizing the numerator of the first term.

Whence P or $v\sqrt{(2\,a\,v-1)} = -3\,a\,.\,u - \int \frac{d\,v}{\sqrt{(2\,a\,v-1)}},$

$$= -3\,a\,.\,u - \frac{1}{a}\sqrt{(2\,a\,v-1)};$$

$$\therefore\ u = -\frac{1}{3\,a}\left(v + \frac{1}{a}\right)\sqrt{(2\,a\,v-1)},$$

$$= -\frac{1}{3\,a}\left(\frac{1}{x^2} + \frac{1}{a\,x}\right)\sqrt{(2\,a\,x-x^2)}.$$

To integrate $\dfrac{d\,\theta}{1-e^2\,.\,\cos^2\theta}$, we may put it in the form

$$\frac{d\,.\,2\,\theta}{2-e^2-e^2\cos 2\,\theta},$$

which is deducible from the integral of the general expression
$\dfrac{d\,\phi}{a+b\,\cos\phi}$; to integrate which, let $\cos\phi = \dfrac{1-x^2}{1+x^2}$, whence

$$d\,\phi = \frac{2}{1+x^2};$$

$$\therefore\ \frac{d\phi}{a+b\,\cos\phi} = \frac{2\,d\,x}{(a+b)+(a-b)\,x^2},$$

$$\text{or} = \frac{2\,d\,x}{(b+a)-(b-a)\,x^2},$$

according as a is greater or less than b.

These last being rational fractions, we have, by the common method, after restoring the value of

$$x = \sqrt{\left(\frac{1-\cos\phi}{1+\cos\phi}\right)} = \tan\frac{\phi}{2},$$

$$\int \frac{d\,\phi}{a+b\,\cos\phi} = \frac{2}{\sqrt{(a^2-b^2)}}\,.\,\tan^{-1}\cdot\left\{\frac{a-b}{\sqrt{(a^2-b^2)}},\,\tan\frac{\phi}{2}\right\},$$

when $a > b$,

and $\int \dfrac{d\phi}{a + b \cos \phi} = \dfrac{1}{\sqrt{(b^2 - a^2)}} \cdot$

$$\log \dfrac{(b + a) \cos \frac{1}{2}\phi + \sqrt{b^2 - a^2} \cdot \sin \frac{1}{2}\phi}{(b + a) \cos \frac{1}{2}\phi - \sqrt{b^2 - a^2} \cdot \sin \frac{1}{2}\phi}, \text{ when } b > a.$$

Now, in the example proposed,

$$a = 2 - e^2, \ b = - e^2, \ \tfrac{1}{2}\phi = \theta;$$

$$\int \dfrac{d\theta}{1 - e^2 \cdot \cos^2\theta} = \dfrac{1}{\sqrt{(1 - e^2)}} \cdot \tan^{-1}\left\{\dfrac{\tan\theta}{\sqrt{(1 - e^2)}}\right\}, \ (e < 1;)$$

$$\int \dfrac{d\theta}{1 - e^2 \cdot \cos\theta} = \dfrac{1}{2\sqrt{(e^2 - 1)}} \cdot \log \dfrac{\sqrt{(e^2 - 1)} \cdot \cos\theta - \sin\theta}{\sqrt{(e^2 - 1)} \cdot \cos\theta + \sin\theta}, (e > 1.)$$

To integrate $x y^2 \, dy + y^3 \, dx = a^3 \, dx$, we have

$$\dfrac{y^2 \, dy}{a^3 - y^3} = \dfrac{dx}{x},$$

$$\therefore \ - \log (a^3 - y^3)^{\frac{1}{3}} = \log \dfrac{x}{c^6};$$

$$\therefore \ (a^3 - y^3) x^3 = c^6.$$

In the next example let $\dfrac{dx}{dt} = p$, $\dfrac{dy}{dt} = q$; then the first equation becomes

$$\dfrac{dp}{dt} = a - b p^2, \text{ or } dt = \dfrac{dp}{a - b p^2};$$

$$\therefore \ t = \dfrac{1}{2\sqrt{ab}} \cdot \log \dfrac{\sqrt{a} + p \sqrt{b}}{\sqrt{a} - p \sqrt{b}};$$

no constant being added, because (as by hypothesis) t and p vanish together.

Making $\sqrt{\dfrac{a}{b}} = m$, and $2\sqrt{(ab)} = k$, the last equation gives

$$\dfrac{p}{m} = \dfrac{e^{kt} - 1}{e^{kt} + 1} \text{ or } dx = \dfrac{e^{kt} - 1}{e^{kt} + 1} \cdot m \, dt;$$

by the integration of which, $2 b x = \log \cdot \dfrac{(e^{kt} + 1)^2}{4 e^{kt}}$;

x being made to vanish with t;

$$\therefore\ e^{2 b x} = \frac{(e^{kt} + 1)^2}{4 e^{kt}}; \therefore\ e^{2 b x} - 1 = \frac{(e^{kt} - 1)^2}{4 e^{kt}};$$

$$\therefore\ \frac{e^{2 b x}}{e^{2 b x} - 1} = \left(\frac{e^{kt} + 1}{e^{kt} - 1}\right)^2 = r^2 \text{ suppose.}$$

From the second of the given equations, we have

$$d t = \frac{1}{b} \cdot \frac{d q}{(c - q)^2}; \therefore\ t = \frac{1}{b c} \cdot \frac{q}{c - q};$$

the integral being so taken that q and t may vanish together. From this last,

$$\frac{d y}{d t} = \frac{b c^2 t}{b c t + 1};$$

whence $e^{b y} (b c t + 1) = e^{b c t}$. $\hspace{2em}(A)$

Now, from the above expression for r we get

$$e^{k t} = \frac{r + 1}{r - 1}, \text{ or } t = \log \left(\frac{r+1}{r-1}\right)^{\frac{1}{k}}; \text{ whence } b c t = \log \left(\frac{r+1}{r-1}\right)^{\frac{b c}{k}},$$

$$e^{b c t} = \left(\frac{r+1}{r-1}\right)^{\frac{b c}{k}}; 1 + b c t = \log e + \log \left(\frac{r+1}{r-1}\right)^{\frac{b c}{k}} = \log e \left(\frac{r+1}{r-1}\right)^{\frac{b c}{k}};$$

and finally by the substitution of these values in equation (A)

$$e^{b y} . \log . e \left(\frac{r + 1}{r - 1}\right)^{\frac{b c}{k}} = \left(\frac{r + 1}{r - 1}\right)^{\frac{b c}{k}}.$$

To integrate $\left(\dfrac{d z}{d x}\right)^2 + \left(\dfrac{d z}{d y}\right)^2 = a^2$;

substitute a^2 for $m^2 - 1$ in the solution of the equation given in page 44 of the *Solutions of the Cambridge Problems for* 1830, and we get the equations

$$\dot{z} - a\,x - \sqrt{(a^2 - a^2)}\,y = \phi\,(a),$$

$$- x - \frac{a\,y}{\sqrt{(a^2 - a^2)}} = \phi'\,(a)\,;$$

from which a is eliminated when an arbitrary form is given to $\phi\,(a)$.

17. If a straight line PSP_1 revolve about a fixed point S; and if perpendiculars PM, $P_1\,M_1$ be drawn upon a fixed straight line passing through S, find the equation to the curve in which

$$\frac{1}{PM^n} + \frac{1}{P_1\,M_1{}^n} = \frac{2}{u^n}.$$

$$\text{Let } \bar{S}P = r, \quad SP_1 = r_1, \quad \frac{1}{y} = u;$$

$$PM = y, \quad P_1M_1 = y_1, \frac{1}{y_1} = u_1:$$

Then, from similar triangles,

$$\frac{r_1}{r} = \frac{y_1}{y} = \frac{u}{u_1}, \text{ or } u_1\,r_1 = u\,r.$$

Let now $r = \phi(u)$, $r_1 = \phi(u_1)$, in which the form of ϕ is to be determined, and first let u_1 be any given function of u, or $u_1 = a\,(u)$.

Then, since $u_1\,r_1 = u\,r$,

$$a\,(u)\,.\,\phi\,a\,(u) = u\,.\,\phi\,(u).$$

Now this equation will evidently be fulfilled, if we can find such a function $f\,(u) = u\,.\,\phi\,(u)$, which will not be changed when $a\,(u)$ is written for u; that is, if we can solve the equation

$$f\,a\,(u) = f\,(u).$$

But, if a be of such a kind that $a\,a\,(u) = u$, this solution will evidently be any equation that involves u and $a\,(u)$ symme-

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trically, because the expressions u and $\alpha\,(u)$ only change place by writing $\alpha\,(u)$ for u:

So that if $\chi\left\{u,\ \alpha\,(u)\right\}$ be taken to denote a symmetrical function of u and $\alpha\,(u)$, we have

$$f(u) = u \cdot \phi(u) = \chi\left\{u,\ \alpha\,(u)\right\},$$

$$\text{or } r = y \cdot \chi\left\{\frac{1}{y},\ \alpha\left(\frac{1}{y}\right)\right\}; \qquad (1)$$

which is the equation representing all the curves which have the required property; but it must be remembered that in the above process no distinction has been made of the cases in which SP, SP_1 are drawn in the same or in opposite directions: but it is easily seen, that in the latter case $\alpha\left(\dfrac{1}{y}\right)$ must be affected with a contrary sign to that of $\dfrac{1}{y}$ in the equation (1).

Thus, in the case proposed, we have

$$r = y \cdot \chi\left\{\frac{1}{y},\ -\sqrt[n]{\left(\frac{2}{a^n} - \frac{1}{y^n}\right)}\right\} \cdot \qquad (2)$$

As the process is instructive, we will shew how this equation may be verified in one or two cases.

Let $n = 1$; then $r = y \cdot \chi\left\{\dfrac{1}{y},\ \left(\dfrac{2}{a} - \dfrac{1}{y}\right)\right\}$,

which expresses a particular class comprised in the order of curves expressed by (1): then, for an individual case let

$$\chi\left\{\frac{1}{y},\ \left(\frac{2}{a} - \frac{1}{y}\right)\right\} = a\left(\frac{1}{y} - \frac{1}{a}\right) = \frac{a}{y} - 1;$$

$\therefore$ the equation is $r = a - y$, which is that of a parabola so situated that the axis of x (or fixed line upon which the perpendiculars PM, $P_1 M_1$ are drawn) passes through the focus in the direction of the latus rectum. To shew how this answers the

conditions, let θ be the angle made by SP with SM; so that $r = \dfrac{y}{\sin \theta}$, which gives

$$y = \frac{a \sin \theta}{1 + \sin \theta},$$

$$\text{and} \therefore y_1 = \frac{a \sin (\pi + \theta)}{1 + \sin (\pi + \theta)}$$

$$= -\frac{a \sin \theta}{1 - \sin \theta};$$

$$\therefore \frac{1}{y} + \frac{1}{y_1} = \frac{1 + \sin \theta}{a \sin \theta} - \frac{1 - \sin \theta}{a \sin \theta}$$

$$= \frac{2}{a}.$$

Now when P comes to the vertex, $\theta = \dfrac{\pi}{2}$, and $SP = PM = \dfrac{a}{2}$; and $P_1 M_1$ is infinite, and $\therefore \dfrac{1}{P_1 M_1} = 0$; $\therefore \dfrac{1}{PM} + \dfrac{1}{P_1 M_1} = \dfrac{2}{a}$; and thus the parabola answers to the conditions in every extreme case.

As another instance let $n = 1$, and

$$\chi \left\{ \frac{1}{y}, \left(\frac{2}{a} - \frac{1}{y} \right) \right\} = 2\,a \left(\frac{1}{y} - \frac{1}{a} \right),$$

$$\text{and} \therefore r = 2\,(a - y) \text{ or } r = \frac{2\,a}{1 + 2 \sin \theta},$$

which represents an hyperbola; the fixed line passing through the focus in the direction of the latus rectum as before.

If $\chi \left\{ \dfrac{1}{y}, \left(\dfrac{2}{a} - \dfrac{1}{y} \right) \right\} = \dfrac{a}{2} \left(\dfrac{1}{y} - \dfrac{1}{a} \right)$ and $\therefore 2\,r = a - y$, or

$$r = \frac{a}{2 + \sin \theta} = \frac{1}{2} \cdot \frac{a}{1 + \frac{1}{2} \cdot \sin \theta};$$

which represents an ellipse in a situation precisely similar to those of the other two conic sections just found.

Problems of this kind having hitherto more rarely occurred than it is probable they will hereafter; and being moreover of a highly interesting nature; we shall make no apology for offering a few more.

(1). Let it be required to find curves in which the sum of any two radii, drawn in opposite directions shall be invariable.

Retaining the same figure and notation, we have

$$\frac{y_1}{r_1} = \frac{y}{r}; \text{ or, making } y = \phi(r), \ y' = \phi(r_1), \ r_1 = a(r);$$

$$\frac{\phi \, a(r)}{a(r)} = \frac{\phi(r)}{r};$$

which equation, like the one in the last problem, is satisfied by such a function $f(r) = \dfrac{\phi(r)}{r}$, such that $f a(r) = f(r)$, of which the solution is a symmetrical function of r and $a(r)$, when $a \, a(r) = r$, which is the case in this problem; we therefore have

$$f(r) \text{ or } \frac{\phi(r)}{r} = \chi\left\{a, \ a(r)\right\}; \ \therefore y = r, \ \chi\left\{r, \ - a(r)\right\}.$$

Now $a(r) = 2a - r$ in this case; and to obtain a particular species of curves let $\chi\left\{r, \ -(2a - r)\right\} = \dfrac{r - (2a - r)}{2a};$

$$\therefore 2ay = r^2 - (2ar - r^2), \text{ or } ay = r^2 - ar.$$

Or, referring to polar co-ordinates,

$$a \sin \theta = r - a; \text{ or } r = a + a \sin \theta;$$

which is the equation to the cardioide.

To verify this, we have $r = a + a \sin \theta,$

$$r_1 = a + a \sin(\pi + \theta),$$

$$= a - a \sin \theta \, ;$$

$$\therefore r + r_1 = 2 \, a,$$

and thus the required property is exhibited.

(2). Let it be required to find curves such that the sums of the reciprocals of any two radii drawn in opposite directions may be invariable.

Let $r = \dfrac{1}{u}$, $r_1 = \dfrac{1}{u_1}$, $u_1 = a(u)$, $y = \phi(u)$;

then $\dfrac{y_1}{y} = \dfrac{r_1}{r} = \dfrac{u}{u_1}$, or $u_1 y_1 = u \, y$;

$$\therefore a(u) \cdot \phi \, a(u) = u \cdot \phi(u) \, ;$$

$$\therefore \text{if } f(u) = u \, , \phi(u), \; f \, a(u) = f(u) \, ;$$

$$\therefore y = r \cdot \chi \left\{ \frac{1}{r}, \, - a \left(\frac{1}{r} \right) \right\}.$$

Let $a \left(\dfrac{1}{r} \right) = \dfrac{2}{a} - \dfrac{1}{r}$;

$$\chi \left\{ \frac{1}{r}, \, - a \left(\frac{1}{r} \right) \right\} = \frac{a \, e}{2} \left\{ \frac{1}{r} - \left(\frac{2}{a} - \frac{1}{r} \right) \right\} = a \, e \left(\frac{1}{r} - \frac{1}{a} \right);$$

$$\therefore y = a \, e - e \, r; \text{ or if } y = r \cos \theta,$$

$$r = \frac{a \, e}{1 + e \cos \theta},$$

which expresses either of the conic sections.

(3). Let it be required to find curves, in which the sum of the squares of any two radii, drawn at right angles to one another, may be invariable.

Let $r = \phi(\theta)$, $r_1 = \phi \left(\dfrac{\pi}{2} + \theta \right)$;

$$\therefore r^2 + r_1^2 = (\phi(\theta))^2 + \left\{ \phi \left(\frac{\pi}{2} + \theta \right) \right\}^2 = 4 \, a^2 \text{ suppose,}$$

or if $\theta = \dfrac{\pi}{2} \cdot z$,

$$\left\{ \phi\left(\dfrac{\pi}{2}z\right) \right\}^2 + \left\{ \phi\,\dfrac{\pi}{2}(z+1) \right\}^2 = 4\,a^2.$$

Now if $\left\{ \phi\left(\dfrac{\pi}{2}z\right) \right\}^2 = u_z$, $\left\{ \phi\,\dfrac{\pi}{2}(z+1) \right\}^2 = u_{z+1}$;

$$\therefore u_{z+1} + u_z = 4\,a^2.$$

Now the equation $u_{z+1} + u_z = 0$ is evidently satisfied by by making $u_z = \cos \pi z$;

$$\text{for } \cos \pi (z+1) + \cos \pi z = 0;$$

hence the complete integral is

$$u_z = 2\,a^2 + C \cos \pi z.$$

This solution would belong only to a particular species of curves if C were merely a constant; but it is evident that the nature of the integration only requires that C shall be some function which does not change by changing u_z into u_{z+1}; and of such a kind is $\cos 2 \pi z$ or any function of it; so that for C we may substitute $f(\cos 2 \pi z)$:

$$\therefore u_z = 2\,a^2 + \cos \pi z \cdot f(\cos 2 \pi z);$$

or, since $u_z = (\phi(\theta))^2 = r^2$, and $\pi z = 2\,\theta$;

$$r^2 = 2\,a^2 + \cos 2\,\theta \cdot f(\cos 4\,\theta).$$

To verify this by some particular instances, first let $f(\cos 4\,\theta) = 0$, and we have $r = \sqrt{(2)}\,a$, which denotes a circle in which all the radii are drawn from the centre which obviously answers the conditions.

Next let $C = 2\,a^2$; then $r^2 = 2\,a^2(1 + \cos 2\,\theta)$, or $r = 2\,a\,.$ $\cos \theta$, which again denotes a circle, but in which the radii are drawn from a point in the circumference; and it is known that if any two chords be drawn at right angles to one another from

any point in the circumference, the sum of their squares will be constantly equal to the square of the diameter.

(4). Let it be required to find curves such that any two ordinates at a given distance (a) shall be cut by the curve at angles, the sum of whose tangents is constant.

Let y, y' be the ordinates at a distance a from each other corresponding to the abscissæ x ; $x' = x + a$:

$$\text{then } \frac{dy}{dx} + \frac{dy'}{dx'} = 2\,m \text{ suppose}:$$

$$\text{let } \frac{dy}{dx} = \phi(x)\,;\,\therefore \frac{dy'}{dx'} = \phi(x + a);$$

$$\therefore \phi(x) + \phi(x + a) = 2\,m,$$

or, if $x = a\,z$, and $\therefore \phi(x) = \phi(a\,z) = u_z$, and $\phi(a(z+1)) = u_{z+1}$,

$$u_z + u_{z+1} = 2\,m\,;$$

from which equation,

$$u_z = m + C\,.\,\cos \pi z\,;$$

where C is either a constant or some function of z which does not change when z becomes $z + 1$:

$$\therefore \frac{dy}{dx} = m + C\,.\,\cos \frac{\pi x}{a}\,;$$

$$\therefore y = m\,x + C \sin \frac{\pi x}{a}\,,\text{ if } C \text{ be constant.}$$

This represents a curve which cuts the straight line $y = m\,x$ at points where $x = 0,\ a,\ 2\,a,\ 3\,a,$ &c.

For more examples see Mr. Herschel's supplementary volume to Lacroix's Differential Calculus.

18. Prove that

$$\frac{d^m u_z}{dx^m} = \left\{\log\,(1 + \Delta)\right\}^m.\,u_z,$$

where any power of Δ as Δ^p, in the expansion of $\{\log (1 + \Delta)\}^m$, when u_x is joined to it, denotes the pth difference of u_x.

And if a_0, a_1, a_2, ... a_n be the co-efficients of $\Delta^m u_x$, $\Delta^{m+1} u_x$, ... $\Delta^{m+n} u_x$ in the expansion of $\{\log (1 + \Delta)\}^m . u_x$; shew that a_n is found in terms of a_{n-1}, &c. from the following equation,

$$n\, a_n = \frac{1}{2} \left\{ n - (m + 1) \right\} a_{n-1} - \frac{1}{3} \left\{ n - 2 . (m + 1) \right\} a_{n-2}$$
$$+ \frac{1}{4} \left\{ n - 3 (m + 1) \right\} a_{n-3} - \&c.$$
$$\pm \frac{1}{n} \left\{ n - (n - 1) . (m + 1) \right\} a_1 \pm \frac{n\, m}{n + 1}.$$

To prove that $\dfrac{d^n u_x}{d\, x^n} = \left\{ \log (1 + \Delta) \right\}^n u_x$. see page 489 of the Translation of Lacroix's Differential and Integral Calculus: and for the latter part of the problem we have only to deduce the co-efficient of any term of the mth power of a multinomial in terms of the rest; for this purpose we will for the present abandon the notation used above and consider the expansion of

$$(1 + c_1 x + c_2 x^2 + \, ... + c_n x^n + . .)^m$$
$$= 1 + a_1 x + a_2 x^2 + \, ... + a_n x^n + . . .$$

To determine a_1, a_2, &c. we take the logarithmic differential of each side of the equation; and by freeing each side of the resulting equation, arranging the terms according to the powers of x, and equating those that are homologous on each side, we get

$$a_1 = m\, c_1,$$
$$2\, a_2 = (m - 1)\, c_1 a_1 + 2\, m\, c_2,$$
$$3\, a_3 = (m - 2)\, c_1 a_2 + (2\, m - 1) . c_2 a_1 + 3\, m\, c_3;$$
. . . .

$$n \, a_n = \left\{ m - (n-1) \right\} c_1 \, a_{n-1} + \left\{ 2\,m - (n-2) \right\} c_2 \, a_{n-2}$$
$$+ \left\{ 3\,m - (n-3) \right\} c_3 \, a_{n-3} + \ldots$$
$$\ldots + \left\{ (n-1)\,m - 1 \right\} c_{n-1} \, a_1 + n\,m\,c_n + \ldots$$

Now the function we have to expand is

$$\left\{ \log (1 + \Delta) \right\}^m = \Delta^m \left\{ 1 - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots \pm \frac{\Delta^n}{n} \right\}$$
$$= a_o \, \Delta^m + a_1 \, \Delta^{m+1} + \ldots + a_n \, \Delta^{m+n}$$

the general co-efficient of which is expressed by the above formula, when we make

$$c_1 = -\frac{1}{2}, \; c_2 = \frac{1}{3}, \; \ldots \; c_n = \pm \, \frac{1}{n+1} \, .$$

For the different methods of expanding any function of a multinomial, see Mr. Arbogast's Calcul des Derivations, Woodhouse's Principles of Analytical Calculation, and a paper by Mr. Knight in the Philosophical Transactions for 1811.

19. Having given u_o, u_a, u_{2a}, three values of a function near its maximum, observed at times 0, a, $2\,a$; find the time when the function will be a maximum.

And if the declinations of the Sun at noon on three successive days were $23^\circ.\,27'$, $23^\circ.\,27.9$, $23^\circ.\,27'.6$, find when the declination was greatest.

In page 105 of Mr. Herschel's Examples we have the formula

$$n = \frac{(\beta^2 - \gamma^2)\, v_a - (a^2 - \gamma^2)\, v_\beta + (a^2 - \beta^2)\, v_\gamma}{2 \left\{ (\beta - \gamma)\, v_a - (a - \gamma)\, v_\beta + (a - \beta)\, v_\gamma \right\}};$$

where n is the time corresponding to the maximum of a function whose observed values at the times a, β, γ are v_a, v_β, v_γ.

When the observations are equidistant and the epoch fixed at the first, $a = 0$, $a - \beta = \beta - \gamma$ or $\gamma = 2\beta$; substituting these values and adopting the notation given in the problem, we have

$$n = \frac{a}{2} \cdot \frac{3\,u_o - 4\,u_a + u_{2\,a}}{u_o - 2\,u_a + u_{2\,a}}.$$

But $a =$ one day,

$$u_o = 23^\circ.27',$$
$$u_a = 23^\circ.27'.9,$$
$$u_{2\,a} = 23^\circ.27'.6;$$
$$\therefore n = \frac{1}{2} \cdot \frac{3}{1.2} \text{ days,}$$
$$= 1 \text{ day } 6 \text{ hours:}$$

∴ the time of the solstice was 6 o'clock in the evening of the second day.

20. If the Earth be an oblate spheroid, and from any point Q above it perpendiculars QM, QN be drawn to the axis and equator respectively, intersecting a meridian in P and p; and if tangents PT, $p\,t$ to this meridian meet the axis and equator in T and t, and if the straight line which joins T and t cuts the meridian in E and F, E and F are the extreme points of the meridian visible from Q.

Taking C for the centre of the elliptic meridian let a, b be the major and minor semi-axes; then, if x', y' be the co-ordinates of any point without the ellipse, from which two tangents are drawn, the equation of the line joining the two points of contact will be (*Hamilton's Analytical Geometry, p.* 116.)

$$a^2 y' \cdot y + b^2 x' \cdot x = a^2 b^2. \qquad (1)$$

If, then, $x' = a$, $y' = \beta$ be the co-ordinates of the point Q, the equation of the line joining the extreme visible points will be

$$a^2 \beta . y + b^2 a . x = a^2 b^2; \qquad (2)$$

∴ if T', t' be the points where this line cuts the axis and equator; we have, by making x and y successively $= 0$, in (2),

$$CT' = \frac{b^2}{\beta}, \quad C\,t' = \frac{a^2}{a}.$$

We next find the points of contact of tangents drawn from T' or from a point where (in equation (1))

$$x' = 0, \quad y' = CT' = \frac{b^2}{\beta};$$

by which (1) becomes $y' = \beta$; ∴ the tangent from T' meets the curve at P: similarly, by making

$$y' = 0, \quad x' = C\,t' = \frac{a^2}{a} \text{ in (1)},$$

we have $x' = a$; ∴ the tangent from t' meets the curve at p:

therefore since tangents from T' and t' meet the curve in P and p, the points T', t' coincide respectively with the points T, t, by the construction, that is, the line joining the extreme points of vision (2) is the line $T\,t$, and since E, F are the only points in which this line meets the curve, they are the extreme points of vision.

Q. E. D.

21. A given hemisphere rests with its base upon a horizontal plane, and a given uniform rod, one end of which is moveable about a horizontal axis fixed in the plane, is placed against the hemisphere so as to be a tangent to a great circle of it; and the rod by its pressure puts the hemisphere in motion; find the

equation for determining the motion of the rod when the friction of the plane varies as the pressure of the hemisphere upon it. And when there is no friction, find the angular velocity of the rod when it comes to the plane.

Let A be the intersection of the rod with the horizontal plane, P the point of its contact with the hemisphere whose centre is at C.

Let $2l$ be the whole length of the rod, m its mass, m' that of the hemisphere, and a its radius: let $AP = r$, $AC = x$, $\angle PAC = \theta$; Pg the pressure at P.

Then, if f be the ratio of the friction to the pressure, the whole horizontal moving force of the hemisphere is

$$P g \sin \theta - f (P g \cos \theta + m' g) = \mu \cdot \frac{d^2 x}{d t^2}; \quad (\mu = m + m').$$

Now Pg is the statical pressure of the rod together with that due to its motion:

$$\therefore P = \frac{l}{r} m \cos \theta + \mu r \frac{d^2 \theta}{d t^2};$$

and the above equation put in the form

$$P g (\sin \theta - f \cos \theta) - f m' g = \mu \cdot \frac{d^2 x}{d t^2},$$

becomes

$$g \left(\frac{l}{r} m \cos \theta + \mu r \cdot \frac{d^2 \theta}{d t^2} \right) (\sin \theta - f \cos \theta) - f m' g = \mu \cdot \frac{d^2 x}{d t^2}.$$

But $x \sin \theta = a$, $x \cos \theta = r$, and $r \tan \theta = a$:

$$\therefore \frac{d x}{d t} = - \frac{a \cos \theta}{\sin^2 \theta} \cdot \frac{d \theta}{d t};$$

$$\therefore \mu \sin \theta \cdot \frac{d^2 x}{d t^2} = \mu a (1 + 2 \cot^2 \theta) \cdot \frac{d \theta^2}{d t^2} + \mu a \cot \theta \cdot \frac{d^2 \theta}{d t^2}$$

$$= g \left(\frac{l}{a} m \sin^2 \theta + \mu\, a \cos \theta \cdot \frac{d^2 \theta}{d t^2} \right) (\sin \theta - f \cos \theta) - f\, m'\, g$$

which is the required equation for determining θ from t.

But for the second part of the problem, we must consider that when the rod has come into such a position that $r = 2\, l$, or $\tan \theta = \dfrac{a}{2\, l}$, it is no longer a tangent to the hemisphere, and the above equation no longer expresses the motion, and must, therefore, be re-constructed.

Let $\angle\ ACP = \phi$; then if $P\, g$ be the pressure at P;

$$P\, g \cdot \cos \phi = m \cdot \frac{d^2 x}{d t^2} :$$

but in the triangle ACP, $x = a\, \dfrac{\sin (\theta + \phi)}{\sin \theta}$;

$$\therefore P\, g \cdot \cos \phi = m\, a \cdot d^2 \cdot \frac{\dfrac{\sin (\theta + \phi)}{\sin \theta}}{d t^2} :$$

also $P\, g \cdot 2\, l \sin (\theta + \phi) = \dfrac{\mu\, g\, l \cdot \cos \theta + \mu\, l \cdot \dfrac{d^2 \theta}{d t^2}}{m\, a \cdot d^2 \cdot \dfrac{\sin (\theta + \phi)}{\sin \theta}}{d t^2}$;

$$\therefore \frac{2\, l\, m\, a}{d t^2} \cdot \frac{\sin (\theta + \phi)}{\cos \phi} \cdot d^2 \frac{\sin (\theta + \phi)}{\sin \theta} = m\, g\, l \cos \theta + \mu\, l \cdot \frac{d^2 \theta}{d t^2}.$$

But $\dfrac{\sin (\theta + \phi)}{\sin \theta} = \cos \phi + \cot \theta \cdot \sin \phi$

$$= \cos \phi + \cot \theta \cdot \frac{2\, l}{a} \cdot \sin \theta$$

$$= \cos \phi + \frac{2\, l}{a} \cdot \cos \theta ;$$

$$\therefore d \cdot \frac{\sin (\theta + \phi)}{\sin \theta} = - \cos \phi\, d \phi - \frac{2\, l}{a} \cdot \sin \theta\, d \theta ;$$

but $\sin \phi = \dfrac{2\,a}{r} \cdot \sin \theta$; $\therefore \cos \phi \cdot d\phi = \dfrac{2\,l}{a} \cdot \cos \theta\, d\theta$;

$$\therefore d\phi = \frac{2\,l}{a} \cdot \frac{\cos \theta}{\cos \phi} \cdot d\theta;$$

$$\therefore d \cdot \frac{\sin (\theta + \phi)}{\sin \theta} = -\frac{2\,l}{a} \left\{\cos \theta \cdot \tan \phi + \sin \theta\right\} d\theta,$$

and $\dfrac{\sin (\theta + \phi)}{\sin \theta} = \sin \theta + \cos \theta \cdot \tan \phi$;

$$\therefore \frac{\sin (\theta + \phi)}{\cos \phi} \cdot d^2 \frac{\sin (\theta + \phi)}{\sin \theta}$$

$$= -\frac{2\,l}{a} (\sin \theta + \cos \theta \cdot \tan \phi)\, d \cdot \left\{(\sin \theta + \cos \theta \cdot \tan \phi)\, d\theta\right\}$$

$$= -\frac{l}{a} \frac{d}{d\theta} \left\{(\sin \theta + \cos \theta \cdot \tan \phi)\, d\theta\right\}^2;$$

$$\therefore C - a^2 (\sin \theta + \cos \theta \cdot \tan \phi)^2 \cdot \frac{d\theta^2}{dt^2} = m\,g\,l \cdot \sin \theta + \frac{\mu\,l}{2} \cdot \frac{d\theta^2}{dt^2}.$$

The constant C is determined by giving to $\dfrac{d\theta}{dt}$ the value found from the first part of the problem when $\theta = 90° - \phi$, or $\sin \theta = \dfrac{a}{2\,l}$; then making $\theta = 0$, and $\phi = 0$, we have the value of $\dfrac{d\theta}{dt}$ at the point required.

22. If normals be drawn at every point of the rhumb line, find the locus of their intersection with the equator, the Earth being considered an oblate spheroid.

The projection of any point of the rhumb line on the equator, and the corresponding point of the locus required evidently lie in the same straight line drawn from the centre of the equator; so that if r be the radius vector of the projection of the rhumb

line, and ρ that of the curve required, the difference of r and ρ must always equal the sub-normal to the point in the elliptic meridian to which r is the abscissa.

Let $z^2 = \dfrac{b^2}{a^2}(a^2 - r^2)$ be the equation of any meridian; then since the sub-normal $= -\dfrac{b^2}{a^2} r$, we must have

$$\rho = r - \frac{b^2}{a^2} r, \text{ or } \rho = e^2 r,$$

we have, therefore, only to find an equation between r and an angle θ made by it with a given diameter of the equator and to substitute in it the value of r in terms of ρ as expressed above.

To find the equation of the rhumb line, we may conceive it to be generated by a point moving uniformly along the meridian, while the meridian revolves uniformly about its axis; then by the composition of motion, the arcs due to the two motions in the same time must be to one another in a constant ratio (m); and therefore their limiting ratio must be the same; so that if s be any arc of the meridian, we have

$$\frac{r\,d\theta}{d s} = m, \text{ or } \frac{r\,d\theta}{d r} \cdot \frac{d r}{d s} = m.$$

But from the equation $z^2 = \dfrac{b^2}{a^2}(a^2 - r^2)$,

$$\frac{d s}{d r} = \sqrt{\left(\frac{a^2 - e^2 r^2}{a^2 - r^2} \right)}; \therefore \frac{d\theta}{d r} = \frac{m}{r} \sqrt{\left(\frac{a^2 - e^2 r^2}{a^2 - r^2} \right)}$$

whence, by integration,

$$\theta + c = m \left\{ \log \sqrt{\left(\frac{u-1}{u+1} \right)} - e \log \sqrt{\left(\frac{u-e}{u+e} \right)} \right\},$$

$$\text{where } u^2 = \frac{a^2 - e^2 r^2}{a^2 - r^2} = e^2 \cdot \frac{a^2 e^2 - \rho^2}{a^2 e^4 - \rho^2}.$$

To determine c we may suppose the generating point to set out from the equator when the meridian begins to revolve; or that $\theta = 0$, when $r = a$; which gives $c = 0$.

23. If x, y, z, r, and x', y', z', r' be the co-ordinates and distances of two planets m and m_1 from the centre of the Sun supposed at rest, and if $\lambda =$ the distance of m from m_1,

$$\text{and } Q = \frac{x\,x' + y\,y' + z\,z'}{r'^3} - \frac{1}{\lambda},$$

prove that the axis-major of the ellipse of curvature at the point (x, y, z) of m's orbit is

$$= \frac{1 + m}{2\,m_1 \int \left(\dfrac{dQ}{dx}\,dx + \dfrac{dQ}{dy}\,dy + \dfrac{dQ}{dz}\,dz \right)},$$

where $1 =$ mass of the Sun.

How does it appear that the mean motions of the planets are subject to no *secular* variations?

By the principles given in page 34 we have the accelerating force of the Sun on m expressed by $\frac{1}{r^2}$, while that of m on the Sun is expressed by $\frac{m}{r^2}$; and therefore the force tending to diminish the distance between the Sun and m must $= \frac{1 + m}{r^2}$, which also expresses the effective force by which m would be urged towards the Sun's centre regarded as fixed and not influenced by the attraction of m; thus the whole effective force on m (abstracting the influence of m_1) varies inversely as the square of the distance from a fixed centre about which it consequently describes an ellipse. Now in estimating the effect of

m_1 in *disturbing* this elliptic motion, we have not merely to consider its accelerating effect $\frac{m_1}{r'^2}$ on m, but the force by which it tends to increase or diminish the distance between the Sun and m_1 which is the measure of what is called the disturbing force; and this also may from the above reasoning be considered as proceeding from the Sun's fixed centre.

To apply this we will resolve all the forces in the direction of one line (the axis of x), considering those positive which act from the Sun towards m. Thus

$$\text{the force of } m_1 \text{ on the Sun} = \frac{m_1 x'}{r'^3},$$

$$\text{the force of } m_1 \text{ on } m = \frac{m_1 (x' - x)}{\lambda^3};$$

$$\therefore \text{ disturbing force} = \frac{m_1 x'}{r'^3} - \frac{m_1 (x' - x)}{\lambda^3},$$

which may be taken with a contrary sign when we consider the Sun as a fixed centre of force; then similarly resolving the forces in the directions of y and z, the equations of motion give

$$\frac{d^2 x}{d t^2} + \frac{(1 + m) x}{r^3} + m_1 \left\{ \frac{x'}{r'^3} - \frac{x' - x}{\lambda^3} \right\} = 0,$$

$$\frac{d^2 y}{d t^2} + \frac{(1 + m) y}{r^3} + m_1 \left\{ \frac{y'}{r'^3} - \frac{y' - y}{\lambda^3} \right\} = 0,$$

$$\frac{d^2 z}{d t^2} + \frac{(1 + m) z}{r^3} + m_1 \left\{ \frac{z'}{r'^3} - \frac{z' - z}{\lambda^3} \right\} = 0.$$

But $Q = \dfrac{x x' + y y' + z z'}{r'^3} - \dfrac{1}{\lambda}$,

$$\lambda = \sqrt{ \left\{ (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \right\} };$$

and $r'^3 = \sqrt{(x'^2 + y'^2 + z'^2)^3}$;

whence we observe that

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$$\frac{x'}{r'^3} - \frac{x' - x}{\lambda^3} = \frac{d\,Q}{d\,x},$$

$$\frac{y'}{r'^3} - \frac{y' - y}{\lambda^3} = \frac{d\,Q}{d\,y},$$

$$\frac{z'}{r'^3} - \frac{z' - z}{\lambda^3} = \frac{d\,Q}{d\,z};$$

and thus the above equations of m's motion become

$$\frac{d^2 x}{d\,t^2} + \frac{(1 + m)\,x}{r^3} + m_1 \frac{d\,Q}{d\,x} = 0,$$

$$\frac{d^2 y}{d\,t^2} + \frac{(1 + m)\,y}{r^3} + m_1 \frac{d\,Q}{d\,y} = 0,$$

$$\frac{d^2 z}{d\,t^2} + \frac{(1 + m)\,z}{r^3} + m_1 \frac{d\,Q}{d\,z} = 0.$$

By eliminating $d\,t$ from the above equations we should arrive at two equations expressing the species, magnitude, and position of the path described by m, which would be an ellipse if m_1 were nothing: but although the forces resulting from m_1 are such that the orbit of m, while subject to their influence, is not an ellipse, they are yet so inconsiderable, compared with the centripetal force of m towards the Sun, that the *excursions* from the ellipse, during any one revolution, never bear any considerable ratio to the dimensions of the ellipse itself. The planet may therefore be supposed to move in a curve in which the co-ordinates at any assigned time are deduced from those of the ellipse by making them to vary; the variations being small spaces or 'errors' due to the disturbing forces. And thus this method of variations, remarkable for its elegance as well as its extensive application in Mathematical Philosophy, enables us to avoid the complexity that would be encountered in at once attacking the question on an hypothesis strictly conformable to nature; by setting out with one more simple whose deviations from the true one it is the business of the Problem to determine.

This does not apply to the variations of the *elements* of the orbit; for although the forces which cause the deviations from the fixed ellipse are very small, their accumulated effect during several revolutions may be such as to bear a very sensible ratio to the dimensions of the ellipse itself. The ellipse therefore from which the above mentioned variations are reckoned must be considered as continually shifting its position and varying in its dimensions, and the next desideratum would be to determine these elements corresponding to any given time. The following method given by Lagrange is theoretically applicable to both kinds of variation, but in the construction of tables it is most convenient to apply it to those of the latter kind only.

If at the expiration of any instant of time the disturbing force should suddenly cease to act, the planet would go on to describe an ellipse of which the arc described during that instant would be an infinitesimal portion; its plane being that passing through the small element and the Sun's centre; the major axis and eccentricity being determined from the curvature and position of this element. Such an ellipse is evidently an *ellipse of curvature* to the actual path of the planet, in which the radius vector, deflection, and velocity at the point of contact, are the same in both curves, and the position and dimensions of this for any given time constitute what are called the *elements* of the planet's orbit. To determine the ellipse of curvature, we take the three equations last found, suppressing those terms multiplied by m_1; multiplying these by $2\,d\,x$, $2\,d\,y$, $2\,d\,z$ respectively, then adding and integrating, we get

$$\frac{1+m}{a} = \frac{2\,(1+m)}{r} - \frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2},$$

a being an arbitrary constant, which from the common elliptic theory is easily seen to be the semi-axis major of the ellipse that would be described without any disturbing force: the value of *a* from this equation is

$$\frac{2(1+m)}{r} - \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\},$$

which corresponds to that of the ellipse of curvature when we give to, r, $\left(\frac{dx}{dt}\right)^2$, $\left(\frac{dy}{dt}\right)^2$, $\left(\frac{dz}{dt}\right)^2$ the values they actually have in the case of disturbed motion.

To find these, we take the three equations first found, and by multiplying the first by $2\,dx$, the second by $2\,dy$, the third by $2\,dz$, and integrating them severally, we get

$$\left(\frac{dx}{dt}\right)^2 = -(1+m)\int \frac{2x\,dx}{r^3} - 2m_1 \int \frac{dQ}{dx}\,dx,$$

$$\left(\frac{dy}{dt}\right)^2 = -(1+m)\int \frac{2y\,dy}{r^3} - 2m_1 \int \frac{dQ}{dy}\,dy,$$

$$\left(\frac{dz}{dt}\right)^2 = -(1+m)\int \frac{2z\,dz}{r^3} - 2m_1 \int \frac{dQ}{dz}\,.\,dz;$$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= \frac{2(1+m)}{r} - 2m_1 \int \left(\frac{dQ}{dx}\,dx + \frac{dQ}{dy}\,dy + \frac{dQ}{dz}\,dz\right):$$

which by being substituted in the above expression for a gives that required.

<div style="text-align: right">Q. E. D.</div>

By applying this process to determine the variation of a, we find by carrying the approximation to the same extent as in determining the perturbations, that its expression contains no terms that are functions of t except such as are of the form $\sin (A + B t)$ or $\cos (A + B t)$, and therefore that the variations of a cannot increase indefinitely as the time increases, and must therefore be subject to periodical maxima and minima: and thus the mean motions which depend entirely on a have no *secular* variation.

24. If a be the mean distance of a planet from the Sun, and $l =$ the length of the line of nodes, then the time of the planet's passage from node to node through the perihelion is

$$= \frac{a^{\frac{3}{2}} p}{\pi} \left\{ \tan^{-1} \sqrt{\frac{l}{2a-l}} - \frac{l}{2a} \cdot \sqrt{\frac{2a-l}{l}} \right\},$$

where $p =$ the periodic time of the Earth about the Sun, and $1 =$ its mean distance from it.

The solution of this problem is immediately deducible from Lambert's Theorem for the case of the ellipse; and as demonstrations of this important property are rarely met with in elementary treatises, we shall commence by presenting one to the reader.

Let r be the radius vector, u the eccentric, and v the true anomaly at a time t; r', u', v' similar quantities corresponding to a time t': we then have the following well-known formulæ :

$$t = a^{\frac{3}{2}} (u - e \cdot \sin u), \qquad t' = a^{\frac{3}{2}} (u' - e \cdot \sin u'),$$

$$r = a (1 - e \cdot \cos u), \qquad r' = a (1 - e \cdot \cos u'),$$

$$r = \frac{a(1-e^2)}{1 + e \cos v}, \qquad r' = \frac{a(1-e^2)}{1 + e \cos v'};$$

$$\therefore t' - t = a^{\frac{3}{2}} (u' - u - e \sin u' + e \sin u).$$

This expresses the time employed by the planet in describing the elliptic arc comprised between the radii r and r'.

If we make $t' - t = 2\theta$, $u' + u = 2s'$, $u' - u = 2s$, observing that

$$\sin u' - \sin u = 2 \sin \frac{u' - u}{2} \cdot \cos \frac{u' + u}{2}$$

$$= 2 \sin s \cdot \cos s',$$

we have

$$\theta = a^{\frac{3}{2}} \, (s - e \, . \, \sin s \, . \, \cos s). \qquad (1).$$

Let $c =$ the chord which joins the extremities of r and r'; $v' - v$ will be the angle contained by these; and considering the triangle formed by these three straight lines, we have

$$c^2 = r^2 + r'^2 - 2 \, r \, r' \, . \, \cos \, (v' - v),$$

an equation which we may write thus

$$(r + r')^2 - c^2 = 2 \, r \, r' \, \{1 + \cos \, (v' - v)\}$$

$$= 4 \, r \, r' \, . \, \cos^2 \frac{v' - v}{2} \, . \qquad (2).$$

We next have to substitute in this equation for the angles v and v' their values in terms of u and u': now comparing the two values of r with each other, we find

$$\sin v = \frac{a \, \sqrt{(1 - e^2)} \, . \, \sin u}{r},$$

$$\cos v = \frac{a \cos u - a \, e}{r};$$

similarly

$$\sin v' = \frac{a \, \sqrt{(1 - e^2)} \, . \, \sin u'}{r'},$$

$$\cos v' = \frac{a \cos u' - a \, e}{r'}.$$

Again, since

$$r \, r' = a^2 \, \{1 - e \, (\cos u + \cos u') + e^2 \cos u \, . \, \cos u'\},$$

observing that

$$\cos \, (v' - v) = \cos v \, . \, \cos v' + \sin v \, . \, \sin v',$$

we have

$$r \, r' \, \{1 + \cos \, (v' - v)\}$$

$$= a^2 \, \{1 + \cos \, (u' - u) - 2 \, e \, (\cos u + \cos u') + e^2 \, (1 + \cos \, (u' + u))\},$$

$$\text{or } r\, r'\,.\, \cos^2 \frac{v'-r}{2}$$

$$= a^2 \left\{ \cos^2 \frac{u'-u}{2} - 2\, e\,.\, \cos \frac{u'-u}{2}\,.\, \cos \frac{u'+u}{2} + e^2\,.\, \cos^2 \frac{u'+u}{2} \right\}$$

$$= a^2 (\cos s - e\,.\, \cos s')^2. \qquad (3)$$

Make $\quad r + r' + c = 2p, \; r + r' - c = 2q;$

which gives

$$(r + r')^2 - c^2 = 4pq \text{ and } r + r' = p + q;$$

by equations (2) and (3) we shall have

$$\sqrt{(pq)} = a\,(\cos s - e\,.\, \cos s').$$

Substituting in the same manner for r and r' their values in terms of u and u' in the equation $p + q = r + r'$, we find

$$p + q = a \left\{ 2 - e\,(\cos u + \cos u') \right\}$$

$$= 2a\,(1 - e \cos s\,.\, \cos s'). \quad (4)$$

If from this equation we find the value of $e\,.\, \cos s'$ and substitute it in the preceding, we have

$$\cos s = \frac{\sqrt{(pq)} + \sqrt{\left\{ (2a - p)\,(2a - q) \right\}}}{2a};$$

$$\therefore \cos 2s = \frac{(a - p)\,(a - q) + \sqrt{\left\{ pq\,(2a - p)(2a - q) \right\}}}{a^2}$$

$$= \left(\frac{a-p}{a}\right)\left(\frac{a-q}{a}\right) + \sqrt{\left\{ \left\{ 1 - \left(\frac{a-p}{a}\right)^2 \right\}\left\{ 1 - \left(\frac{a-q}{a}\right)^2 \right\} \right\}}.$$

If we then suppose

$$\frac{a-q}{a} = \cos z, \; \frac{a-p}{a} = \cos z',$$

we have

$$\cos 2s = \cos z\,.\, \cos z' + \sin z\,.\, \sin z' = \cos(z' - z),$$

which gives

$$\tan s = \tan \frac{z' - z}{2} = \frac{\sin z' - \sin z}{\cos z' + \cos z}.$$

Let us now resume the equation (1) which expresses the time employed by the planet in describing the elliptic arc subtended by the chord c. If we eliminate $e \cos s'$ by means of equation (4), we find

$$\theta = a^{\frac{3}{2}} \left\{ s - \left(\frac{2a - p - q}{2a} \right) . \tan s \right\}, \qquad (5)$$

an equation which no longer contains e.

If we substitute for s and $\tan s$ their values, observing that

$$\frac{2a - p - q}{a} = \cos z + \cos z',$$

we have

$$t' - t = a^{\frac{3}{2}} (z' - z - \sin z' + \sin z);$$

which is Lambert's Theorem.

For the problem given above we shall use the equation (5).

The line of nodes is that chord which passes through the focus, and which is consequently equal to the sum of the two radii vectores which bound the arc : we therefore have from the equations

$$r + r' + c = 2p, \ r + r' - c = 2q,$$

$l = p$ and $q = 0$; and consequently $\cos s = \sqrt{\dfrac{2a - l}{2a}}$,

and $\therefore \tan s = \sqrt{\dfrac{l}{2a - l}}$:

$$\therefore \theta = a^{\frac{3}{2}} \left\{ \tan^{-1} \sqrt{\frac{l}{2a - l}} - \frac{2a - l}{2a} . \sqrt{\frac{l}{2a - l}} \right\}$$

$$= a^{\frac{3}{2}} \left\{ \tan^{-1} \sqrt{\frac{l}{2a - l}} - \frac{l}{2a} \sqrt{\frac{2a - l}{l}} \right\}.$$

But if T be the time of describing this arc, $T = (t' - t) \sqrt{m}$ (m being the sun's absolute force) or $T = 2\theta \sqrt{m}$; but since m is nearly constant for every planet,

$$p = \frac{2\pi}{\sqrt{m}} \text{ or } \sqrt{m} = \frac{p}{2\pi};$$

$$\therefore T = \frac{p\theta}{\pi} = \frac{a^{\frac{3}{2}} p}{\pi} \left\{ \tan^{-1} \sqrt{\frac{l}{2a-l}} - \frac{l}{2a} \cdot \sqrt{\frac{2a-l}{l}} \right\}.$$

<div align="right">Q. E. D.</div>

FRIDAY EVENING.

1. FIND all the solutions in positive whole numbers of the equation $11x + 15y = 1031$.

Reducing $\frac{11}{15}$ to a continued fraction, and deducing from thence a set of converging fractions, we find the last two of these to be $\frac{3}{4}$ and $\frac{11}{15}$; whence, by the property of these fractions,

$$15 \times 3 - 11 \times 4 = 1;$$

$$\therefore 15 \times 3093 - 11 \times 4124 = 1031,$$

or $15 \times 3093 - 11 \times 4124 + 11 \times 15m - 11 \times 15m = 1031$, supposing m to be any positive whole number.

This last equation, being so arranged as to coincide with the proposed equation, becomes

$$11(15m - 4124) + 15(3093 - 11m) = 1031;$$

whence, by comparison of the terms,

$$x = 15m - 4124,$$
$$y = 3093 - 11m.$$

<div align="center">K</div>

But since x and y are restricted to positive values, we must have

$$m > \frac{4124}{15} > 274 + \text{remainder};$$

$$m < \frac{3093}{11} < 281 + \text{remainder};$$

therefore the extreme values of m are 275 and 280; and consequently

those of x are 1, 16, 31, 46, 61, 76, 91 :

those of y are 68, 57, 46, 35, 24, 13, 2.

2. Determine which is the greatest term of the expansion of $(a + b)^n$.

Let N_p denote the pth term; and let $\dfrac{b}{a} = v$; then

$$N_p = \frac{n(n-1)\ldots(n-p+2)}{1.2.3\ldots(p-1)} a^{n-p+1} b^{p-1},$$

$$N_{p+1} = \frac{n(n-1)\ldots(n-p+1)}{1.2.3\ldots p} a^{n-p} b^p;$$

$$\therefore \frac{N_p}{N_{p+1}} = \frac{p}{(n-p+1)v}.$$

If for any value of p the expression last found be greater than 1; it appears that it will be so for every succeeding value; that is, if N_p be greater than N_{p+1}, the remaining terms will be a decreasing series; and an increasing one in the contrary case; the greatest term therefore will correspond to the least value of p that makes the above expression greater than 1.

If then N_p be the greatest term, we must have

$$\frac{p}{(n-p+1)v} > 1, \text{ or } p > (n-p+1)v;$$

$$\therefore p > (n+1) \cdot \frac{v}{v+1};$$

and as the least value of p which fulfils this condition is the one required, and p must be a whole number, it must be that whole number which is next greater than

$$(n+1)\frac{v}{v+1} \quad \text{or} \quad (n+1) \cdot \frac{b}{a+b} \cdot$$

3. Shew that $\sqrt{5}$ is greater than $\dfrac{682}{305}$ and less than $\dfrac{2889}{1292}$, and that it differs from the latter fraction by a quantity less than $\dfrac{1}{2 \times 305 \times 1292} \cdot$

If a set of converging fractions be deduced from a continued fraction, it is well known that their several values are alternately less and greater than the true value of the continued fraction; and that the difference between this true value (a) and any converging fraction is less than the difference between (a) and the converging fraction immediately preceding.[*]

If then $\dfrac{P}{P'}$, $\dfrac{Q}{Q'}$ be any two successive fractions converging to the value of (a), we have

$$\frac{Q}{Q'} - a < a - \frac{P}{P'},$$

$$\text{or } \frac{2Q}{Q'} - 2a < \frac{Q}{Q'} - \frac{P}{P'};$$

$$\therefore \frac{Q}{Q'} - a < \frac{1}{2}\left(\frac{Q}{Q'} - \frac{P}{P'}\right).$$

To verify this in the proposed example, we have

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[*] See Garnier's Elemens d'Algebre, p. 473, or Barlow's Theory of Numbers.

$$\sqrt{5} = 2 + \frac{\sqrt{5} - 2}{1} = 2 + \frac{1}{\sqrt{5} + 2},$$

$$\sqrt{5} + 2 = 4 + \frac{\sqrt{5} - 2}{1} = 4 + \frac{1}{\sqrt{5} + 2};$$

consequently, since $\sqrt{5} + 2$ recurs, we have

$$\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}} \ ad \ infinitum.$$

The converging fractions deduced from this are

$$\frac{2}{1}, \ \frac{9}{4}, \ \frac{38}{17}, \ \frac{161}{72}, \ \frac{682}{305}, \ \frac{2889}{1292}, \ \&c.$$

$$\therefore \sqrt{5} > \frac{682}{305} \text{ and } < \frac{2889}{1292}.$$

Also, from the above general condition, we have, by substituting the values of P, P', Q, Q',

$$\frac{2889}{1292} - \sqrt{5} < \frac{1}{2} \left(\frac{2889}{1292} - \frac{682}{305} \right) < \frac{1}{2 \times 305 \times 1292},$$

$$\text{since } \frac{2889}{1292} - \frac{682}{305} = \frac{1}{305 \times 1292}.$$

Q. E. D.

4. *BC*, *CD* are two consecutive arcs of a parabola, the sagitta of which, bisecting the chords, and parallel to the axis, are equal; prove that the chord of *BCD* is parallel to the tangent at *C*.

Bisect the chords *BC*, *CD* in the points *M*, *N*; and from *B*, *M*, *N*, *D* let the straight lines *B b*, *M m*, *N n*, *D d* be drawn parallel to the axis and meeting a tangent at *C* in the points *b*, *m*, *n*, *d*; let *M m*, *N n* intersect the parabola in the points *P*, *Q*.

By this construction Cb, Cd are bisected in m, n, and Bb, Dd are respectively double of Mm, Nn (Euclid VI. 2).

The sagitta $QN = Nn - Qn$

$$= \frac{1}{2} Dd - Qn;$$

similarly, the sagitta $PM = \frac{1}{2} Bb - Pm$,

and $QN = PM$ by hypothesis;

$$\therefore \frac{1}{2} Dd - Qn = \frac{1}{2} Bb - Pm,$$

or $Bd - 2Pm = Dd - 2Qn.$ (1).

But, by the common property of the parabola, (Hustler's Conic Sections, Prop. XI.)

$$\frac{Pm}{Qn} = \frac{Cm^2}{Cn^2} = \frac{Cb^2}{Cd^2};$$

also since B, D are points in the parabola,

$$\frac{Cb^2}{Cd^2} = \frac{Bd}{Dd}; \quad \therefore \frac{Pm}{Qn} = \frac{Bd}{Dd};$$

$$\therefore \frac{2PM}{Bd} = \frac{2Qn}{Dd};$$

$$\therefore \frac{Bd - 2PM}{Bd} = \frac{Dd - 2Qn}{Dd};$$

$$\therefore (1) \ Bd = Dd;$$

whence, Bb being parallel to Dd, by construction, the figure $BDdb$ is a parallelogram; therefore BD is parallel to bd.

Q. E. D.

5. If a, b, c, be the lengths of the chords of three arcs of a circle, which together make up a semi-circumference, and r the radius of the circle, then
$$4r^3 - (a^2 + b^2 + c^2)\,r - abc = 0.$$

Let $2A$, $2B$, $2C$ be the angles subtended at the centre by the chords a, b, c; then

$$a = 2r \cdot \sin A, \ b = 2r \cdot \sin B, \ c = 2r \cdot \sin C;$$

$$\therefore a^2 + b^2 + c^2 = 4r^2 (\sin^2 A + \sin^2 B + \sin^2 C). \quad (1)$$

But $\qquad\qquad 2A + 2B + 2C = \pi;$

$$\therefore \cos(2A + 2B) + \cos 2C = 0,$$

or $\cos 2A \cdot \cos 2B - \sin 2A \cdot \sin 2B + \cos 2C = 0,$

or $(1 - 2\sin^2 A)(1 - 2\sin^2 B)$

$+ \ 1 - 2\sin^2 C - 4\sin A \cdot \sin B \cdot \cos A \cdot \cos B = 0,$

or $1 - (\sin^2 A + \sin^2 B + \sin^2 C)$

$+ \ 2\sin A \cdot \sin B (\sin A \cdot \sin B - \cos A \cdot \cos B) = 0;$

$\therefore$ (since $\sin A \cdot \sin B - \cos A \cdot \cos B = -\cos(A+B) = -\sin C,$)

$$\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \cdot \sin B \cdot \sin C$$

$$= 1 - \frac{abc}{4r^3}.$$

Whence, by substitution in equation (1),

$$a^2 + b^2 + c^2 - 4r^2 \left(1 - \frac{abc}{4r^3}\right) = 0,$$

$$\text{or, } 4r^3 - (a^2 + b^2 + c^2)r - abc = 0.$$

Q. E. D.

6. If two roots of the cubic $x^3 - qx + r = 0$ be $a + b\sqrt{-3}$ and $a - b\sqrt{-3}$, then will

$$-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = (b - a)^3.$$

The co-efficient of x^2 being nothing, the sum of the roots must be nothing; therefore the remaining root is $-2a$: we then have, from the known properties of the co-efficients,

$$r = 2\,a\,(a^2 + 3\,b^2), \quad q = 3\,(a^2 - b^2);$$

$$\therefore \frac{r^2}{4} = a^2\,(a^2 + 3\,b^2)^2 = a^6 + 6\,a^4\,b^2 + 9\,a^2\,b^4,$$

$$\text{and } \frac{q^3}{27} = (a^2 - b^2)^3 = a^6 - 3\,a^4\,b^2 + 3\,a^2\,b^4 - b^6;$$

$$\therefore \frac{r^2}{4} - \frac{q^3}{27} = 9\,a^4\,b^2 + 6\,a^2\,b^4 + b^6;$$

$$\therefore \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = 3\,a^2\,b + b^3;$$

$$\text{and } \frac{r}{2} = a^3 + 3\,a\,b^2;$$

$$\therefore -\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = b^3 - 3\,b^2\,a + 3\,b\,a^2 - a^3$$

$$= (b - a)^3.$$

<div align="right">Q. E. D.</div>

7.　Required the conditions to be satisfied that in the division of $x^4 + q\,x^2 + r\,x + s$ by $x^2 + a\,x + b$, the remainder may be nothing independently of the value of x, after three terms of the quotient are obtained: and when these conditions are satisfied, obtain an equation for finding b from q, r, s, given, and shew how to solve it.

Since the co-efficient of x^3 is nothing in the dividend, the quotient must be of the form $x^2 - a\,x + b'$;

$$\therefore x^4 + q\,x^2 + r\,x + s = (x^2 + a\,x + b)\,(x^2 - a\,x + b')$$
$$= x^4 + (b + b' - a^2)x^2 + (a\,b' - a\,b)x + b\,b',$$

which equation is supposed to be true independently of the value of x; therefore by the comparison of the homologous terms we have

$$q = b + b' - a^2,$$
$$r = a(b' - b)$$
$$s = b\,b';$$

whence, by the elimination of a and b', we get the equation

$$b^6 - q\,b^5 - s\,b^4 + (2\,q\,s - r^2)\,b^3 - s^2\,b^2 - q\,s^2\,b + s^3 = 0$$

for determining b, which, by multiplying the roots by $\dfrac{1}{\sqrt{s}}$, becomes

$$b^6 - \frac{q}{\sqrt{s}}\,.\,b^5 - b^4 + \frac{2\,q\,s - r^2}{s\,\sqrt{s}}\,.\,b^3 - b^2 - \frac{q}{\sqrt{s}}\,.\,b + 1 = 0,$$

a recurring equation of six dimensions which may therefore be resolved into two cubics.

8. Assuming that if δp, δq, δr be the virtual velocities of three forces P, Q, R, which keep a point at rest, $P\,\delta p + Q\,\delta q + R\,\delta r = 0$, in whatever direction the virtual motion of the point takes place; prove that the forces are proportional to the sides of a triangle drawn in their directions.

Let the directions of the forces P, Q, R make angles a, β, γ with a fixed straight line; and let k be the extent of the virtual motion of the point, its direction making an angle θ with the fixed line: then, by the definition of a virtual velocity,

$$\delta p = k\,.\,\cos(\theta - a),$$
$$\delta q = k\,.\,\cos(\theta - \beta),$$
$$\delta r = k\,.\,\cos(\theta - \gamma),$$

which values substituted in the equation

$$P\,\delta p + Q\,\delta q + R\,\delta r = 0$$

give

$$P\,.\,\cos(\theta - a) + Q\,.\,\cos(\theta - \beta) + R\,.\,\cos(\theta - \gamma) = 0,$$

or $P \cdot \cos \alpha + Q \cdot \cos \beta + R \cdot \cos \gamma$

$+ (P \sin \alpha + Q \sin \beta + R \sin \gamma) \cdot \tan \theta = 0.$

But this equation is, by the supposition, to be true independently of the value of θ; whence

$$P \cdot \cos \alpha + Q \cdot \cos \beta + R \cdot \cos \gamma = 0,$$
$$P \cdot \sin \alpha + Q \cdot \sin \beta + R \cdot \sin \gamma = 0.$$

By multiplying the first of these equations by $\sin \gamma$, the second by $\cos \gamma$, and subtracting, we get

$$P \cdot \sin (\gamma - \alpha) + Q \cdot \sin (\gamma - \beta) = 0;$$
$$\therefore \frac{P}{Q} = \frac{\sin (\gamma - \beta)}{\sin (\alpha - \gamma)};$$

similarly, $\dfrac{P}{R} = \dfrac{\sin (\beta - \gamma)}{\sin (\alpha - \beta)},$

$$\frac{Q}{R} = \frac{\sin (\alpha - \gamma)}{\sin (\beta - \alpha)};$$

that is, P, Q, R are as the sides of a triangle whose angles are $\beta - \gamma$, $\alpha - \gamma$, and, $\beta - \alpha$, which are the angles made by (Q, R), (P, R), and (P, Q).

<div align="right">Q. E. D.</div>

9. A ladder rests with its foot on a horizontal plane, and its upper extremity against a vertical wall; having given its length, the place of its centre of gravity, and the ratios of the friction to the pressure on the plane and on the wall, find its position when in a state bordering upon motion.

Referring the whole to rectangular co-ordinates of which x is horizontal and y vertical; let the extremity A of the ladder rest against the wall, and the extremity B against the plane; let R, mR be the pressure and friction at B, and R', nR'

those at A; let the ladder make an angle θ with the wall, and G being its centre of gravity, let $AG = a$, $BG = b$.

Now, if forces X, Y act in the directions of the axes of x and y, applied at a point whose co-ordinates are x, y, the well known conditions of equilibrium give the equations

$$\Sigma(X) = 0, \ \Sigma(Y) = 0, \ \Sigma(Yx) - \Sigma(Xy) = 0.$$

Now the whole weight W of the ladder acting vertically at G; and R, R' acting perpendicularly to the plane and wall, while mR, nR' act along these, we have

$$\Sigma(X) = R' - mR = 0,$$
$$\Sigma(Y) = nR' + R - W = 0,$$
$$\Sigma(Yx) - \Sigma(Xy) = R(a+b).\sin\theta - Wa.\sin\theta - R'(a+b)\cos\theta = 0.$$

The two first of these equations give

$$R' = mR, \ R = \frac{W}{mn+1};$$

which values, substituted in the third equation, give, after reduction,

$$\tan\theta = \frac{m(a+b)}{b - mn.a};$$

or if, $b = ca$, $\tan\theta = \frac{m(c+1)}{c - mn}$,

which is independent of either the weight or length of the ladder.

10. Find the content of the solid generated by the revolution of the curve whose equation is

$$(a^2 + x^2)y^2 - x^2(a^2 - x^2) = 0$$

about the axis of x.

The content

$$= \pi \int y^2 \, dx = \pi \int dx \cdot \frac{x^2(a^2 - x^2)}{a^2 + x^2},$$

$$= \pi \int d x \cdot \left\{ 2 a^2 - \frac{2 a^4}{a^2 + x^2} - x^2 \right\},$$

$$= \pi \left\{ 2 a^2 x - 2 a^3 \cdot \tan^{-1} \frac{x}{a} - \frac{x^3}{3} \right\}, \left(\begin{matrix} x = - a \\ x = + a \end{matrix} \right)$$

$$= \left(\frac{10}{3} - \pi \right) \pi a^3.$$

11. The straight line joining any points P and Q of the surface of an ellipsoid is bisected by a plane passing through the centre of the ellipsoid, and through the line of intersection of the tangent planes at P and Q.

Let the co-ordinate planes be a system of conjugate diametral planes, one $(x\,y)$ bisecting, and another $(y\,z)$ parallel to PQ, so that if

x, y, z be the co-ordinates of P,

$x, y, - z$ will be the co-ordinates of Q;

therefore if, a, b, c be the three semi-conjugate diameters coinciding with the axes of x, y, z; the equation of the tangent plane at P will be

$$\frac{x\,x'}{a^2} + \frac{y\,y'}{b^2} + \frac{z\,z'}{c^3} = 1 ;$$

that at Q $\quad \dfrac{x\,x'}{a^2} + \dfrac{y\,y'}{b^2} - \dfrac{z\,z'}{c^2} = 1.$

Now at the intersection of these two tangent planes, the values of x', y', z' must be the same in each equation; therefore by taking the sum of these equations and clearing each side of the resulting equation of the common factor 2, we get

$$\frac{x\,x'}{a^2} + \frac{y\,y'}{b^2} = 1,$$

which is one of the equations of the intersection of the tangent planes : to get the other, we co bine this last with either of the two preceding and thus find it to be $z' = 0$, which shews that the intersection of the tangent planes at P and Q lies in the plane of x, y ; that is, the straight line PQ is bisected by a plane passing through the centre of the ellipsoid, and through the intersection of the tangent planes at P and Q.

<div align="right">Q. E. D.</div>

A similar process would readily shew that an analogous property exists in each of the conic sections; viz. that the straight line joining any two points P, Q, is bisected by a straight line drawn through the centre, (or parallel to the axis in the case of the parabola) and through the point of intersection of the tangent at P and Q.

12.   A curve surface is described by a straight line always passing through two straight lines, the equations to which are $x = a$, $y = b$; and $x = a'$, $z = b'$; and through a curve $z = f(y)$, in the plane of zy ; shew that the equation of the surface is

$$\frac{x b' - a' z}{x - a'} = f\left(\frac{x b - a y}{x - a} \right).$$

Let the equations of the generating line be

$$\left. \begin{array}{l} y = m x + a \\ z = n x + \beta \end{array} \right\} \qquad (1)$$

in which, m, n, a, β are variables depending on the different positions of the line, and are to be determined so as to satisfy the given conditions.

Since the line (1) is to pass through points where $z = f(y)$, $x = 0$, or where $y = a$, $z = \beta$; the surface generated by it must have for its equation $\beta = f(a)$,

or, since $\beta = z - n\,x$, and $\alpha = y - m\,x$,

$$z - n\,x = f'(y - m\,x). \qquad (2)$$

But the line (1) is also to pass through points where $x = a$, $y = b$, and $x = a'$, $z = b'$; and therefore the equations

$$y - b = m\,(x - a)$$
$$z - b' = n\,(x - a')$$

must be true in all positions of this line; from which

$$m = \frac{y - b}{x - a}, \quad n = \frac{z - b'}{x - a'};$$

which values being substituted in equation (2) give

$$\frac{x\,b' - a'\,z}{x - a'} = f\left(\frac{x\,b - a\,y}{x - a}\right).$$

Q. E. D.

13. When a very small conical pencil of rays is reflected at a spherical surface, the transverse section of the reflected pencil will be a circle, at a distance from the point of reflection, which is an harmonic mean between the distances of the focal lines in the primary and secondary planes from the same point.

Let AB, CD be the axes of the elliptic section of the pencil at the surface of the mirror, O being the centre: let m be the point of intersection of rays reflected at C, D; and n that of rays reflected at A, B. Let Cm, Dm be produced to meet a line $cn\,d$ parallel to CD in the points c, d; then $c\,d$ is a focal line in the secondary plane; and a line $\alpha\,m\,\beta$ parallel to AB and meeting $A\,n$, $B\,n$ in α and β is a focal line in the primary plane.

Between the focal lines take two points e, f in $A\,n$, $B\,n$

such that ef may be perpendicular to nf, or to ne since the pencil is small; and let

$$AB = a, \ \angle BAa = \theta, \ Om = f_1, \ On = f_2, \ nf = x;$$

then
$$x : ef :: f_2 : a \sin \theta;$$

$$\therefore ef = \frac{a\,x}{f_2} \sin \theta.$$

Take also in mc, md, two points g, h and let gh be at the same distance from n as ef.

Now $mn = f_2 - f_1$, $\therefore mg = f_2 - f_1 - x$.

But
$$mg : gh :: mC : CD,$$

$$\text{or } \ f_2 - f_1 - x : gh :: f_1 : a \sin \theta;$$

$$\therefore gh = (f_2 - f_1 - x)\,\frac{a \sin \theta}{f_1}.$$

Let $gh = ef$, and the transverse section is then a circle;

$$\therefore \frac{a\,x}{f_2} \sin \theta = \frac{a}{f_1}(f_2 - f_1 - x) \sin \theta;$$

$$\therefore f_1 x = f_2^2 - f_1 f_2 - f_1 x,$$

$$\text{or } x = \frac{f_2(f_2 - f_1)}{f_2 + f_1};$$

therefore the distance of this section from O

$$= f_2 - \frac{f_2(f_2 - f_1)}{f_2 + f_1},$$

$$= \frac{2.f_1 f_2}{f_1 + f_2},$$

which is an harmonic mean between f_1 and f_2.

Q. E. D.

14. Integrate the differential equations,

$$\frac{d^2 y}{d x^2} + m \frac{d y}{d x} + n^2 y = A \sin n\,x, \text{ and } \frac{z - b'}{x - a'} = \frac{d z}{d x} + \frac{d z}{d y}\frac{y - b}{x - a};$$

and obtain a particular integral of

$$\frac{d^2 z}{d y^2} + \frac{2 \, d z}{d x} \cdot \frac{d^2 z}{d x \, d y} + \left(\frac{d z^2}{d x^2} - a^2 \right) \frac{d^2 z}{d x^2} = 0.$$

To integrate

$$\frac{d^2 y}{d x^2} + m \frac{d y}{d x} + n^2 y = A \sin n x,$$

let $y = e^{kx}$, k being determined from the equation

$$k^2 + m k + n^2 = 0;$$

and if k, k' be the roots of this, we have, by the process given in page 11 of the Solutions of the Cambridge Problems of 1830, the formula

$$y = c \cdot e^{kx} + c' \cdot e^{k'x} + A \cdot \frac{e^{kx} \int e^{-kx} \sin n x \cdot d x - e^{k'x} \int e^{-k'x} \sin n x \cdot d x}{k - k'};$$

whence, by performing the integrations indicated in the expression,

$$y = c \, e^{kx} + c' \, e^{k'x} + \frac{A}{(k^2 + n^2)(k'^2 + n^2)}$$

$$\left\{ (k \, k' - n^2) \sin n x + n (k + k') \cos n x \right\}.$$

But, from the equation

$$k^2 + m k + n^2 = 0, \; k \, k' - n^2 = 0, \; k + k' = - m,$$
$$\text{and } (k^2 + n^2)(k'^2 + n^2) = m^2 \, n^2;$$

therefore by substitution,

$$y = c \, e^{kx} + c' \, e^{k'x} - \frac{A}{m \, n} \cos n x.$$

To integrate $\dfrac{z - b'}{x - a'} = \dfrac{d z}{d x} + \dfrac{d z}{d y} \cdot \dfrac{y - b}{x - a}$,

which is of the form $P \, p + Q \, q = R$,

where $P = 1$, $Q = \dfrac{y - b}{x - a}$, $R = \dfrac{z - b'}{x - a'}$,

we have the equations

$$P \, d \, y - Q \, d \, x = 0$$
$$P \, d \, z - R \, d \, x = 0 \, ;$$

and the integral of the first being denoted by a, and that of the second by β, the integral required is $\quad \beta = \phi(a)$.

But in the proposed example,

$$P \, d \, y - Q \, d \, x = d \, y - \frac{y - b}{x - a} \, d \, x = 0 \, ;$$

$$\therefore a = \log \left(\frac{y - b}{x - a} \right),$$

$$\text{similarly} \quad \beta = \log \left(\frac{z - b'}{x - a'} \right) \, ;$$

$$\therefore \log \left(\frac{z - b'}{x - a'} \right) = \phi \left\{ \log \left(\frac{y - b}{x - a} \right) \right\} \, ;$$

or, since the form of ϕ is arbitrary,

$$\frac{z - b'}{x - a'} = \phi \left\{ \frac{y - b}{x - a} \right\}.$$

To integrate $\dfrac{d^2 z}{d \, y^2} + \dfrac{2 \, d \, z}{d \, x} \cdot \dfrac{d^2 z}{dx \, dy} + \left(\dfrac{d \, z^2}{d \, x^2} - a^2 \right) \dfrac{d^2 z}{d \, x^2} = 0,$

see Mr. Peacock's Examples, page 477.

15. Eliminate the arbitrary function by differentiation from the equation $\dfrac{z}{x} = \phi \left(\dfrac{y}{x}, \dfrac{x}{t} \right)$; and prove that the integral of every differential equation of the first order and degree between four variables, is of the form $P = f(Q, R)$.

Let $\dfrac{z}{x} = P$, $\dfrac{y}{x} = Q$, $\dfrac{x}{t} = R$; then the equation is

$$P = \phi(Q, R) \, ;$$

then in general

$$\frac{d P}{d x} = \phi'(Q) \cdot \frac{d Q}{d x} + \phi'(R) \cdot \frac{d R}{d x},$$

$$\frac{d P}{d y} = \phi'(Q) \cdot \frac{d Q}{d y} + \phi'(R) \cdot \frac{d R}{d y},$$

$$\frac{d P}{d t} = \phi'(Q) \cdot \frac{d Q}{d t} + \phi'(R) \cdot \frac{d R}{d t},$$

from which three equations the functions $\phi'(Q)$, $\phi'(R)$ may be eliminated. This process is general for all equations of the first degree and order between four variables; but in the proposed example the following process, substantially the same as that above, is shorter.

Since $z = x \cdot \phi \left(\frac{y}{x}, \frac{x}{t} \right)$,

$$\frac{d z}{d x} = \phi \left(\frac{y}{x}, \frac{x}{t} \right) - \frac{y}{x} \phi' \left(\frac{y}{x} \right) + \frac{x}{t} \phi' \left(\frac{x}{t} \right),$$

$$\frac{d z}{d y} = \phi' \left(\frac{y}{x} \right)$$

$$\frac{d z}{d t} = - \frac{x^2}{t^2} \cdot \phi' \left(\frac{x}{t} \right);$$

whence $\dfrac{d z}{d x} = \dfrac{z}{x} - \dfrac{y}{x} \cdot \dfrac{d z}{d y} - \dfrac{t}{x} \cdot \dfrac{d z}{d t}$,

or $\quad x \dfrac{d z}{d x} + y \dfrac{d z}{d y} + t \dfrac{d z}{d t} = z.$

Let $u = 0$ be an equation between the four variables z, y, x, t, in which we may consider z as a function of y, x, t; and y, x as a function of t, the variations of y and x being independent of each other.

These relations may be thus expressed:

$$d'(u) = \frac{d u}{d z} \cdot d z + \frac{d u}{d y} \cdot d y + \frac{d u}{d x} \cdot d x + \frac{d u}{d t} \cdot d t = 0. \quad (1)$$

M

But $\quad dz = \dfrac{dz}{dy}\,dy + \dfrac{dz}{dx}\,dx + \dfrac{dz}{dt}\,dt,$

which being substituted in (1) gives

$$\left(\frac{du}{dz}\cdot\frac{dz}{dy} + \frac{du}{dy}\right)dy + \left(\frac{du}{dz}\cdot\frac{dz}{dx} + \frac{du}{dx}\right)dx$$

$$+ \left(\frac{du}{dz}\cdot\frac{dz}{dt} + \frac{du}{dt}\right)dt = 0;$$

which, on account of the independence of dy and dx, may be separated into the three equations

$$\left.\begin{aligned} \frac{du}{dz}\cdot\frac{dz}{dt} + \frac{du}{dt} &= 0, \\[1mm] \frac{du}{dz}\cdot\frac{dz}{dx} + \frac{du}{dx} &= 0, \\[1mm] \frac{du}{dz}\cdot\frac{dz}{dy} + \frac{du}{dy} &= 0. \end{aligned}\right\} \qquad (2)$$

Let now $\dfrac{dz}{dt} + L\dfrac{dz}{dx} + M\dfrac{dz}{dy} = N,$

be the differential equation of the first order and degree of which $u = 0$ is the primitive equation; by substituting in this the values of $\dfrac{dz}{dt},\ \dfrac{dz}{dx},\ \dfrac{dz}{dy}$ found from equations (2), we get

$$\frac{du}{dt} + L\frac{du}{dx} + M\frac{du}{dy} + N\frac{du}{dz} = 0;$$

whence $\dfrac{du}{dt} = -L\dfrac{du}{dx} + M\dfrac{du}{dy} - N\dfrac{du}{dz},$

which, being substituted in (1), gives

$$d(u) = (dx - L\,dt)\frac{du}{dx} + (dy - M\,dt)\frac{du}{dy} + (dz - N\,dt)\frac{du}{dz} = 0,$$

which will be verified by the conditions

$$d\,x - L\,d\,t = 0,$$
$$d\,y - M\,d\,t = 0,$$
$$d\,z - N\,d\,t = 0.$$
$$\left.\begin{array}{c}\\\\\\\end{array}\right\} \quad (3)$$

Now $d\,(u)$ being $= 0$ in consequence of the conditions (3), the expression u can only consist of certain arbitrary constants; and these constants are obtained by the integration of the equations (3): supposing them to be a, b, c; then the integrals of the equations (3) will determine any three of the variables x, y, z, t in terms of the fourth; and by substituting the values of any three so found in the expression u, the fourth must disappear of itself, and u will thus be of the form $\varphi\,(a,\,b,\,c)$.

If then P, Q, R be the expressions for a, b, c in terms of x, y, z, t obtained from the equations (3), u must be of the form $\phi\,(P,\,Q,\,R)$;

$$\therefore \phi\,(P,\,Q,\,R) = 0,$$
$$\text{or } P = f\,(Q,\,R).$$

Q. E. D.

As an exercise in this integration, the student may take the differential equation obtained at the beginning of this article by the elimination of the arbitrary function, remembering that three other equations may be obtained from the equations (3) by combining any two of them together.

16.   Prove that the integral of $\Delta^3 u_x + a\,\Delta^2 u_x + b\,\Delta u_x + c\,u_x = 0$, is $u_x = C\,\alpha^x + C'\,\beta^x + C''\,\gamma^x$, if $\alpha - 1$, $\beta - 1$, $\gamma - 1$, be the roots of $z^3 + a\,z^2 + b\,z + c = 0$. Also by varying the parameters, obtain the integral of $\Delta^3 u_x + a\,\Delta^2 u_x + b\,u_x + c\,u_x = A_x$.

Let $u_x = \alpha^x$, a form which will satisfy the proposed equation when the constant α is properly determined; for by substituting this form for u_x in the equation it becomes, after clearing it of the factor α^x,

$$(a - 1)^3 + a \cdot (a - 1)^2 + b \cdot (a - 1) + c = 0,$$

an equation for determining a, which may be written thus

$$z^3 + a z^2 + b z + c = 0,$$

if the three values of z denote those of $a - 1$, which values, for the sake of distinguishing them, we denote by $a - 1$, $\beta - 1$, $\gamma - 1$.

We thus have three expressions a^x, β^x, γ^x which severally satisfy the proposed equation; and by connecting these into one expression by means of three distinct arbitrary constants we get the complete integral which is

$$u_x = C a^x + C' \beta^x + C'' \gamma^x. \qquad (1)$$

This will likewise be the integral of the more general equation

$$\Delta^3 u_x + a \Delta^2 u_x + b \Delta u_x + c u_x = A_x,$$

if C, C', C'', instead of constants, be considered functions of x: to determine the form of these, we take the difference of equation (1), which gives

$$\Delta u_x = C (a - 1) a^x + C' (\beta - 1) \beta^x + C'' (\gamma - 1) \gamma^x$$
$$+ a^{x+1} \cdot \Delta C + \beta^{x+1} \cdot \Delta C' + \gamma^{x+1} \cdot \Delta C'';$$

which is reduced to

$$\Delta u_x = C (a - 1) a^x + C' (\beta - 1) \beta^x + C'' (\gamma - 1) \gamma^x, \qquad (2)$$

by supposing

$$a^{x+1} \cdot \Delta C + \beta^{x+1} \cdot \Delta C' + \gamma^{x+1} \cdot \Delta C'' = 0. \qquad (3)$$

We are evidently at liberty to make the hypothesis expressed by (3) with regard to the functions C, C', C'', and one more in addition, since the forms of any two of them are indeterminate and may be any whatever, provided that of the third be determined accordingly: and therefore in taking the difference of (2) we shall make a similar hypothesis with respect to the terms involving ΔC, $\Delta C'$, $\Delta C''$.

We thus have

$$\Delta^2 u_x = C (\alpha - 1)^2 \alpha^x + C' (\beta - 1)^2 \beta^x + C'' (\gamma - 1)^2 \gamma^x, \quad (4)$$

$$\alpha^{x+2} \Delta C + \beta^{x+2} \Delta C' + \gamma^{x+2} \Delta C'' = 0. \quad (5)$$

Again, taking the difference of (4), and reducing by the equations (3), (5), there results

$$\Delta^3 u_x = C (\alpha - 1)^3 \alpha^x + C' (\beta - 1)^3 \beta^x + C'' (\gamma - 1) \gamma^x$$
$$+ C \alpha^{x+3} . \Delta C + C' \beta^{x+3} . \Delta C' + C'' \gamma^{x+3} . \Delta C''. \quad (6)$$

The substitution of the values of Δu_x, $\Delta^2 u_x$, $\Delta^3 u_x$, found from equations (2), (4), (6) in the proposed equation, account being taken of the conditions

$$\alpha^{x+3} + a . \alpha^{x+2} + b . \alpha^{x+1} + c . \alpha^x = 0,$$
$$\beta^{x+3} + a . \beta^{x+2} + b . \beta^{x+1} + c . \beta^x = 0,$$
$$\gamma^{x+3} + a . \gamma^{x+2} + b . \gamma^{x+1} + c . \gamma^x = 0 ;$$

and the other reductions being made, gives

$$\alpha^{x+3} . \Delta C + \beta^{x+3} . \Delta C' + \gamma^{x+3} . \Delta C'' = A_x. \quad (7)$$

We now determine ΔC, $\Delta C'$, $\Delta C''$ from the equations (3), (5), (7); and thus obtain

$$\Delta C = \frac{A_x}{(\beta - \alpha)(\gamma - \alpha) . \alpha^{x+1}},$$

$$\Delta C' = \frac{A_x}{(\alpha - \beta)(\gamma - \beta) . \beta^{x+1}},$$

$$\Delta C'' = \frac{A_x}{(\alpha - \gamma)(\beta - \gamma) . \gamma^{x+1}};$$

$$C = \Sigma . \frac{A_x}{(\beta - \alpha)(\gamma - \alpha) \alpha^{x+1}} + C_1,$$

$$C' = \Sigma . \frac{A_x}{(\alpha - \beta)(\gamma - \beta) \beta^{x+1}} + C_2,$$

$$C'' = \Sigma . \frac{A_x}{(\alpha - \gamma)(\beta - \gamma) \gamma^{x+1}} + C_3;$$

C_1, C_2, C_3 being arbitrary constants.

The same process evidently applies to similar equations of higher orders; and for one of the nth order we might, as in this example, make $n - 1$ arbitrary hypotheses respecting the functions ΔC, $\Delta C'$, &c.

17. Supposing the Earth to be spherical, and the matter in its interior to be compressed according to the law $p = k\,(\rho^2 - \delta^2)$, p being the pressure and ρ the density at any distance r from the centre, and δ the density at the surface; shew that $\rho \propto \dfrac{\sin q\,r}{r}$, q being a certain constant.

The well known laws of the equilibrium of fluids give the equation

$$d p = \rho\,(X\,d x + Y\,d y + Z\,d z),$$

in which if R denote the force acting on any molecule at the distance r from the centre or origin of co-ordinates,

$$X = - \frac{R}{r}\,d x, \;\; Y = - \frac{R}{r}\,d y, \;\; Z = - \frac{R}{r}\,d z;$$

$$\therefore \; d p = - \frac{R \rho}{r}\,(x\,d x + y\,d y + z\,d z)$$

$$= - R \rho\, d r.$$

Now since the molecule at the distance r from the centre is only affected by the attractions of the others situated within that distance, R is the attraction of a sphere of radius r on a particle at its surface; and the density varying only with the distance from the centre, the attraction of this sphere is expressed by its mass multiplied by the inverse square of its radius; therefore, since the mass

$$= \int 4\,\pi\,\rho\,r^2\,d r, \;\; R = \frac{4\,\pi}{r^2} \int \rho\,r^2\,d r;$$

$$\therefore \frac{dp}{dr} = -\frac{4\pi\rho}{r^2} \int \rho\, r^2\, d\,r.$$

But $p = k\,(\rho^2 - \delta^2)$; $\;\therefore \frac{dp}{dr} = 2\,k\,\rho\,\frac{d\rho}{dr};$

$$\therefore 2\,k\,\rho\,.\,\frac{d\rho}{dr} = -\frac{4\pi\rho}{r^2} \int \rho\, r^2\, d\,r,$$

$$\text{or } \frac{d\rho}{dr} + \frac{2\pi}{k\,r^2}.\int \rho\, r^2\, d\,r;$$

therefore differentiating on the supposition of $d\,r$ being constant
dividing the result by r^2, and making $\frac{2\pi}{k} = q^2$;

$$\frac{d^2\rho}{dr^2} + \frac{2}{r}.\frac{d\rho}{dr} + q^2 \cdot \rho = 0.$$

To integrate this, let $\rho = \dfrac{u}{r}$, and by substitution we get

$$\frac{d^2 u}{dr^2} + q^2 u = 0;$$

whence $u = A \sin (q\,r + B),$

A and B being arbitrary constants;

$$\therefore \rho = A \;.\; \frac{\sin (q\,r + B)}{r};$$

but since the density at the centre is not supposed to be infinite,
we must have $B = 0$;

$$\therefore \rho \propto \frac{\sin . \,q\,r}{r}.$$

<div align="right">Q. E. D.</div>

18.   The path of a ship in a horizontal plane, is
referred to rectangular axes, that of x being in the
direction of the wind, and its velocity in sailing
from one given point to another is assumed to be
$f\left(\dfrac{d\,y^2}{d\,x^2}\right)$: required the nature of the brachystochronous
path between the given points.

The expression for the time is $\int_x^\cdot \frac{\sqrt{1+p^2}}{f(p^2)}$; and therefore the general equation

$$N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c. = 0,$$

(*Airy's Tracts*, p. 165).

is reduced to $\frac{d(P)}{dx} = 0$, or $P = C$;

$$\therefore \frac{d}{dp} \cdot \frac{\sqrt{1+p^2}}{f(p^2)} = C,$$

or $\frac{p}{f(p^2)\sqrt{1+p^2}} - \frac{2p \cdot f'(p^2)}{(f(p^2))^2} \cdot \sqrt{1+p^2} = C,$

which equation, as it contains only p and constants, can only denote a system of straight lines, whatever be the form of $f(p^2)$.

To shew how these lines fulfil the conditions of the problem, let θ denote the angle made by the ship's course with the wind, so that $p = \tan\theta$; and the above equation becomes

$$\frac{\sin\theta}{f(\tan^2\theta)} - \frac{2\sin\theta \cdot f'(\tan^2\theta)}{\cos^2\theta \cdot (f'(\tan^2\theta))^2} = C,$$

which exhibits a remarkable property, which is, that, if it be satisfied by any angle θ, it will also be satisfied by the angle $\pi - \theta$; whence it follows that if the straight line joining the fixed points A, B, make an angle θ with the wind which satisfies the above equation, the ship's course will be in the straight line AB throughout: if otherwise, the ship's course will make the angle θ with the wind, as determined above, until she arrive at a point P such that PB shall make an angle with the wind supplementary to θ; when, making a tack, the remainder of her course will be in the straight line PB.

19. If the material particles m, m', ... of any system in motion, pass from the positions a, a', ... to b, b', ... during the very small time δt, and c, c', ... be the positions they would have had, if during δt only the impressed forces had acted; then forces proportional to and in the directions of $\overline{bc}$, $\overline{b'c'}$, ... will produce equilibrium with the pressures on the fixed points and axes of the system.

Any material particle whatever, whether it be single or form part of a system, when in motion and subject to the action of a constant or variable accelerative force, may for a short time δt be supposed to describe a portion of a parabola, because the accelerative force during that time may be considered uniform in direction and magnitude. Suppose a particle m (fig. 1) to be moving at a in the direction ax at the end of t, with a velocity which during the small succeeding interval δt would carry it through ax. Let it actually describe the small parabolic arc ab. Join xb. This line may be taken to represent in magnitude and direction the effective accelerating force acting on the particle. If we call this force F, $xb = \dfrac{F \cdot (\delta t)^2}{2}$. Suppose that if the particle had become free when at a, it would have described the parabolic arc ac. Join xc. This line will represent in magnitude and direction the impressed force on the same scale that xb represents the effective force. For if we call the impressed force f, $xc = \dfrac{f \cdot (\delta t)^2}{2}$. Now the force which compounded with the impressed force will give the effective force, is the force which is represented in magnitude and direction by cb. The forces cb, $c'b'$, ... are supplied by the tensions acting on the particles m, m', &c. If no points of the system be fixed then their tensions exactly destroy each other, because

no motion can result from the mutual action of the parts of the system on each other. But if there be fixed points, these will in general sustain stresses, which may be considered impressed forces, and corresponding to them the tensions $c\,b$, $c'\,b'$, &c. are effective forces. Hence if forces represented in magnitude and direction by $b\,c$, $b'\,c'$, &c. be applied, they will produce equilibrium with the stresses on the fixed points.

Corollary. Suppose γ (fig. 2) to be any position different from b, which the conditions of the system will allow the particle m to take.

$$(c\,\gamma)^2 = (c\,b)^2 + (b\,\gamma)^2 - 2\,(b\,c)\,.\,(b\,\gamma)\cos c\,b\,\gamma.$$

Hence

$$\Sigma\,.\,m\,(c\,b)^2 = \Sigma\,m\,.\,(c\,\gamma)^2 - \Sigma\,.\,m\,.\,(b\,\gamma)^2 + 2\,\Sigma\,.\,m\,(b\,c)\,.\,(b\,n).$$

But it has been shewn that the particles m, m', &c. will be at rest in the positions b, b', &c. if acted upon only by the forces $b\,c$, $b'\,c'$, &c. and if also the pressures on the fixed points remain the same as they actually are when the particles arrive at b, b', &c. And as γ, γ', &c. are positions which the particles may also take, $b\,n$, $b'\,n'$, &c. will be virtual velocities of the forces $b\,c$, $b'\,c'$, &c. Hence because the virtual velocities of the pressures on the fixed points are 0, we have by the principle of virtual velocities,

$$\Sigma\,.\,m\,(b\,c)\,.\,(b\,n) = 0,$$
$$\therefore \Sigma\,.\,m\,(c\,b)^2 = \Sigma\,m\,.\,(c\,\gamma)^2 - \Sigma\,.\,m\,(b\,\gamma)^2,$$
$$\text{or } \Sigma\,.\,m\,(c\,b)^2 < \Sigma\,m\,(c\,\gamma)^2.$$

This is Gauss' Theorem, which he enunciates by saying that in constrained motion the particles take positions differing in the least possible manner that the conditions of the system will admit of, from the positions they would have had if the motion had been free.

20.   Also, if $e, e', \ldots$ be the places $m, m', \ldots$ would have had, if the impressed motions had been compounded with uniform motions during ∂t along the actual paths with the velocities at $a, a', \ldots$; then forces proportional to and in the directions of $\overline{b\,e}, \overline{b'\,e'}, \ldots$ will keep the system at rest; and if $\gamma, \gamma', \ldots$ be any other positions compatible with the conditions of the system $\Sigma \cdot m\,\overline{e\,b^2} < \Sigma \cdot m\,\overline{e\,\gamma^2}$.

In this case $x\,b$ (fig. 3) as before represents the effective force ; af is taken $= a\,x$, xf is joined. $x f$ will be ultimately perpendicular both to $a\,x$ and $a f$. $f\,e$ represents the impressed force, and $a\,e$ is the path which would result from compounding the impressed motion with the uniform motion af along the actual path $a\,b$.   Now $x\,b$ is resolvable into two forces $x\,f, f\,b$, of which fb represents the effective accelerative force *in the direction of the motion*.   But by D'Alembert's Principle the effective forces, such as $f\,b$, applied in the contrary direction, together with the impressed forces, produce equilibrium.   And the resultant of bf and fe is $b\,e$.   Hence the forces represented in magnitude and direction by $(b\,e)$, $(b'\,e')$, &c. will produce equilibrium.

It may be shewn precisely as in the corollary of the preceding question, that if γ, γ', be other positions compatible with the conditions of the system,

$$\Sigma \cdot m\,(e\,b)^2 < \Sigma \cdot m\,(e\,\gamma)^2.$$

Corollary.   If $e\,c$ (fig. 4) be drawn parallel and equal to $x f$, it will be seen that the force $b\,e$ is compounded of $b\,c$ and $c\,e$. $c\,e$ which is the same as $x\,f$ is the part of the effective force *perpendicular* to the direction of the motion and is equal to the centrifugal force.   Hence $b\,e$, the force which must be com-

pounded with the effective force in the direction of the motion, to produce the impressed force, is compounded of the force of tension and the centrifugal force.

21.   An elastic chord $A\,a\,b\,c\,B$ is stretched between two fixed points, A, B; the portion $a\,b\,c$ is made to assume the form of two straight lines $a\,b$, $b\,c$, the points a and c being in the straight line joining A, B, and b at a small distance from it: when the chord is suddenly left to itself, what motion will take place, and what will happen when the motion reaches the fixed points?

In page 379 of Whewell's Dynamics we have

$$\left(\frac{d^2 y}{d\,t^2}\right) = \frac{F\,a\,g}{W}\left(\frac{d^2 y}{d\,x^2}\right)$$

for the equation of the motion of any elastic chord drawn through a small extent from its position of rest; where a, W, F are respectively the length, weight, and tension of the chord, and g gravity.

Making $\dfrac{F\,a\,g}{W} = b^2$, the integral of the above equation is

$$y = F(x + b\,t) + f(x - b\,t), \qquad (1)$$

which exhibits all the varieties of form that the string will take in consequence of its motion corresponding to the different values of t.

Since each of the terms $F(x + b\,t)$, $f(x - b\,t)$ will separately satisfy the partial differential equation, let us consider the effect of taking each of them separately as the expression for y.

Now the equation $y = F(x + b\,t)$ evidently represents a curve of the same form and dimensions as the equation $y = F(x)$

but having each of its ordinates stationed at a distance $b\,t$ on the axis of x from the equal and corresponding ordinate in the latter; supposing the origin to be fixed.

If then in the equation $y = F(x + b\,t)$, t be supposed to increase continuously from nothing to any proposed magnitude, this equation will indicate a *translation* of the curve $y = F(x)$ in a negative direction along the axis of x; and by supposing x and t to vary while y remains constant, we may determine the velocity with which any given ordinate moves.

In fact by making $d(y) = 0$, which gives $d(x + b\,t) = 0$, we have $\dfrac{d\,x}{d\,t} = -b$, which shews that every ordinate and consequently the whole figure moves, in the manner of a wave, along the string with a uniform velocity equal to b. In considering in the same manner the equation $y = f(x - b\,t)$, we find by making $d(x - b\,t) = 0$, or $\dfrac{d\,x}{d\,t} = b$, that figure denoted by $y = f(x)$ moves along the string with an equal velocity and in a direction contrary to that of the figure $y = F(x)$. From this it appears that the whole motion defined by the equation (1) is such as would result from the simultaneous motions of portions of the string with equal and uniform velocities in contrary directions and whose forms and dimensions are defined by the equations $y = F(x)$, $y = f(x)$.

The forms of the functions denoted by F and f being entirely arbitrary, we will consider what will result from supposing them equal and similar, so that equation (1) may take the form

$$y = \frac{1}{2}\left\{f(x + b\,t) + f(x - b\,t)\right\} \qquad (2)$$

which defines an undulatory motion resulting from two equal and similar undulations moving in contrary directions, and

whose forms and dimensions are each defined by the equation

$$y = \frac{1}{2} \cdot f(x).$$

The origin of x and of t being entirely arbitrary, the consideration of either of these quantities as positive or negative when reckoned from a certain origin, is clearly the same as that of its being greater or less than a certain quantity reckoned from a different origin; so that in what has been hitherto said, t has not been supposed to commence with the motion, nor x to commence at the extremity of the string which is considered to be of indefinite length and without any regard to fixed points.

Now, when t is positive, the motion of the string, as defined by equation (2) consists of the translation of the two similar figures above-mentioned in contrary directions, and we observe that when $t = 0$, this equation becomes $y = f(x)$, which shews that at this instant the two figures unite so as to form one in which each of the ordinates is double of the corresponding one in either of the two separate figures; after which the two figures, resuming their forms and dimensions recede with the same uniform velocity from each other to an indefinite distance.

We thus prove that two undulations of the string may cross one another without destroying or at all affecting each other's motions; and the same is obviously true if we give to y the form

$$F(x + bt) + f(x - bt) + F_1(x + bt) + f_1(x - bt) + \&c.$$

since each or any one or any number of these terms severally or collectively satisfy the equation

$$\left(\frac{d^2 y}{d t^2} \right) = b^2 \left(\frac{d^2 y}{d x^2} \right);$$

and therefore any number of undulations, however different in form or magnitude, may proceed at the same time along the same string.

Resuming the case of the two equal undulations in opposite directions, we next enquire under what circumstances they may so affect a point in the string that it may never leave its point of rest; in this case we must have $y = 0$ in equation (2) whatever be the value of t; which condition gives

$$f(x + b\,t) = -f(x - b\,t),$$

shewing that the two component figures of undulation must be on opposite sides of the axis of x in order that they may cross each other in such a manner that a point in the string common to each of them may not be moved. Now since a point is at rest during the whole of the motion, the circumstances of the motion cannot be at all altered by supposing it to be fixed in the first place even in such a manner that the connection between the two parts of the string on each side of it may be broken; and as the motion in the one part of the string cannot affect that in the other, it is clear that the motion in either part will be the same if the other part be altogether removed. It follows therefore that an undulation moving towards a fixed point in any elastic string, will on arriving at that point be reflected so that in its return its figure will be inverted or vertically under the string if it were vertically above it before reflection.

Let us next suppose that there are two fixed points, or, what amounts to the same, that a definite straight line is fixed at its extremities; and let two similar and equal undulations be moving towards each other on the same side of the straight line; we have shown that at a certain instant they will collapse and form one figure in which the height of any point above the level straight line is double that of a corresponding one in either of the separate figures: after thus crossing each other, each will continue to move in its own direction until it reaches one of the fixed points, at which it will be reflected and its figure inverted; the two inverted figures always moving with the same constant velocity b will meet and collapse at the same distance from the

second fixed point as they did from the first when on the opposite side of the figure; and thus after crossing, encountering the fixed points, and again being reflected and inverted, they will be in the situation at which we first considered them and consequently after collapsing in the same place as at first, all the different circumstances of the motion will recur continually, provided that the string be perfectly flexible and subject to no extraneous disturbance.

Now, since a certain constant figure results periodically from the union of the two undulations (or rather semi-undulations) and the form of this figure is entirely arbitrary provided its *contour* do not extend but to a very small distance from the straight line of equilibrium of the string; it is clear that when any such figure is given to the string at rest, motions of the same kind must take place as if the figure had resulted from the union of two semi-undulations as above described; it being remembered that the velocity with which the motion begins to be propagated is the same as after any assigned time.

The solution of the problem enunciated above results immediately from what has been said. Let the length $AB = l$, b representing the velocity of propagation as determined above; then it appears that the point a of the figure $a\,b\,c$ will be depressed until the triangle $a\,b\,c$ (supposing the points b, c joined by an imaginary line) is separated into two triangles such that the height of any point in their sides is half that of a corresponding point in the sides $a\,b$, $b\,c$. These triangles will move each with the velocity b towards the points A, B respectively, at which their figures will be reflected, and will unite forming the inverted figure a', b', c' similar and equal to $a\,b\,c$ and at the same distance from B that the figure $a\,b\,c$ was from A. To find the time we have only to consider that the two semi-undulations must each have to move through a space equal to the

whole length of the string $AB = l$ with a velocity $= b$; the time occupied by this movement is therefore $= \dfrac{l}{b}$, and consequently at the end of another interval equal to this the string will be in its initial state, and so on. Thus the times from the beginning of the motion at which the string will be in its initial state are $\dfrac{2b}{l}$, $\dfrac{4b}{l}$, ... $\dfrac{2mb}{l}$.

22. Let τ be the mean interval between successive passages of the Moon over the meridian of a place in lat. λ, and δ be her declination at a time t reckoned from the high-water of a known spring tide; let τ', δ', be corresponding quantities for the Sun: prove, on the supposition that each luminary causes vertical isochronous oscillations of the ocean, which vary slowly in extent, that the height of the tide at that time above its mean height, is nearly

$$M \cos^2 (\lambda - \delta) \cos \frac{4 \pi t}{\tau} + S \cos^2 (\lambda - \delta') \cos \frac{4 \pi t}{\tau'}.$$

The oscillations caused by each luminary being isochronous, its force accelerating the rise or fall of the tide must be as the height above the mean height; so that if this height be denoted by y, the equation expressing this motion will be

$$\frac{d^2 y}{d t^2} + n^2 y = 0;$$

n^2 being a constant; which, being multiplied by $2 \, d y$, becomes, $d t$ being constant,

$$2 . \frac{d y}{d t} . d \left(\frac{d y}{d t} \right) + 2 n^2 y \, d y = 0,$$

or, by integration,

$$\frac{d y^2}{d t^2} + n^2 y^2 = n^2 a^2;$$

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a being an arbitrary constant, expressing the value of y when $\frac{dy}{dt} = 0$, or, which is the same thing, expressing the greatest value of y caused by one luminary: again,

$$n\,dt = -\frac{dy}{\sqrt{a^2 - y^2}}; \ \therefore nt = \cos^{-1}\frac{y}{a}, \text{ or } y = a\,.\cos nt;$$

no other arbitrary constant being added, because, by hypothesis, y is to be a maximum or equal to a when $t = 0$.

To determine n, we suppose that it will be high-water when $t = 0$, and at the end of every interval equal to $\frac{\tau}{2}$ afterwards; that is that

$$y = a, \text{ or } \cos nt = \cos 2\pi, \text{ when } t \text{ equals } \frac{\tau}{2}, \ \tau, \ 2\ \tau, \ \&c.$$

$$\therefore n\,.\frac{\tau}{2} = 2\pi, \text{ or } n = \frac{4\pi}{\tau}; \ \therefore y = a\,.\cos\frac{4\pi t}{\tau}.$$

The arbitrary quantity a which expresses the maximum elevation for any oscillation is not a constant, but as it is supposed to vary very slowly with respect to t, it may for a short time (as a day) be considered constant provided its true value for the beginning of that time be determined. To determine the form of a, (using the figure in Newton's Principia, prop. 66) let A be the point in the meridian $ABCD$ of the surface of the fluid over which the disturbing body at S is vertical; P the place whose latitude is λ, so that the $\angle PTA = \lambda - \delta$; then the force by which a portion of the fluid at P is drawn from the surface in the direction TS is as PK or $\cos(\lambda - \delta)$ nearly, when S is very distant; this force resolved in the vertical direction TP is as $\cos^2(\lambda - \delta)$; therefore if M denote the elevation at A, that at P will be $M\cos^2(\lambda - \delta)$ which is what y becomes when $\cos\frac{4\pi t}{\tau} = 1$, and is therefore equal to a;

$$\therefore y = M\cos^2(\lambda - \delta)\cos\frac{4\pi t}{\tau}.$$

By reasoning precisely in the same manner, for the other disturbing body which causes an elevation S at the place over which it is vertical, and reckoning t from the same origin as in the above case, so that $\dfrac{4\pi t}{\tau}$, $\dfrac{4\pi t}{\tau'}$ may both be zero at the same time that t is, which corresponds to the time of the spring tide; the elevation caused by the other luminary will be

$$S \cos^2 (\lambda - \delta') \cos \frac{4\pi t}{\tau'};$$

therefore, it being a well-known principle that both oscillations will co-exist, the whole elevation will be nearly

$$M \cos^2 (\lambda - \delta) \cos \frac{4\pi t}{\tau} + S \cos^2 (\lambda - \delta) \cos \frac{4\pi t}{\tau'}.$$

Q. E. D.

The above hypothesis, the truth of which is confirmed by observations, is the result of an investigation given by Laplace in the Memoirs of the Academy for the years 1775, 1776, and 1790, the substance of which articles was subsequently embodied in Book IV. of the Mecanique Celeste.

The hypotheses adopted by Newton, Daniel Bernoulli, and Maclaurin amounted to supposing the earth at rest, and the waters to be under the influence of a stationary attracting body by which they acquired the form necessary for their equilibrium; so that when the rotation of the earth was taken into account, it was supposed to have no other effect than merely to change the situation of any point on its surface with respect to the fluid shell which inclosed it, the configuration of the shell with respect to the attracting body always remaining the same, and its points of greatest elevation always following the motion whether real or apparent of this body so as always to have one of them vertically under it and the other on the opposite side of the earth's surface.

This, however, was totally inadequate to the explanation of one of the principal phenomena, namely that of the almost insensible difference between the heights of any two tides of the same day which, according to this theory, ought to be considerable, whatever be the declinations of the Sun or Moon.

D'Alembert, in his *Reflexions sur la cause des Vents*, calculated the effects of the Sun and Moon on our atmosphere and determined synthetically the oscillations of a fluid of small depth covering a stationary planet, and afterwards endeavoured to determine the oscillations for the case when the disturbing body moves uniformly about a parallel to the equator, and thus arrived, in a very ingenious manner, at the true equations of the motion, but the difficulty of integrating them obliged him to have recourse to suppositions which rendered the solution uncertain.

Laplace, in the first of the three Memoirs above mentioned, has shewn that the effects of the earth's rotation in changing the positions of the particles of the fluid are of the same order as those immediately caused by the actions of the Sun and Moon.

23. Prove that a column of air in a cylindrical pipe of length l, closed at one end, may be made to vibrate so that at distances 2λ, $4\lambda \ldots 2k\lambda$ from the closed end, the air will be stationary, and at distances λ, 3λ, 5λ, $\ldots (2k+1)\lambda$ the last of which $= l$, the density will be constant.

For the solution of this we cannot do better than refer the reader to page 278 of Moseley's Hydrostatics, or, for a complete discussion of the subject, to a very ingenious paper by

Mr. Challis, in Vol. III. part I. of the Cambridge Philosophical Transactions.*

24. What properties of a medium in which the density varies as the pressure, correspond to and serve to explain the following observed properties of light: (1) Its rectilinear and uniform transmission; (2) the different intensity of different rays; (3) the difference of intensity of the same ray at different distances from its origin; (4) the difference which the eye distinguishes in rays by colour; (5) their crossing in all possible ways without mutual disturbance; (6) the interference of two rays, the paths of which nearly coincide in direction and differ in length by a multiple of a certain interval?

(1). If we conceive an agitation to be communicated to a small portion of such a medium, and in a certain direction, the result is a propagation of the agitation, which from the proximity of the particles composing the medium, must take place in every direction; but chiefly in that direction in which the agitation was communicated, and in the opposite direction; those in the transverse directions being less considerable in any degree, according to the constitution of the medium, so that in the luminiferous medium they may be insensible. The agitation which may thus be supposed to take place in one direction in the luminiferous ether will represent the phenomena of a *single*

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* It was intended to give in this place, a historical sketch of the progress of the Problem of Sound and of Vibrating Chords, but the Author is compelled, for want of time, to defer it to the next series of Solutions of the Cambridge Problems.

ray; and the propagated motion so characterised may therefore be defined by the equation

$$\frac{d^2 v}{d\,t^2} = a^2 \cdot \frac{d^2\,v}{d\,x^2},$$

$v = a\,s$, (Moseley's Hydrostatics, p. 269.) where v is the actual velocity of any particle of the medium at a distance x from some fixed origin, s the condensation, at a time t, and a a constant.

The first of the above equations gives, by integration,

$$v = F\,(x - a\,t) + f(x + a\,t),$$

which, as we have shewn in Problem 21, indicates two uniform motions of undulation in opposite directions which co-exist, the velocity of propagation in each being expressed by the constant a.

(2). The different intensity of different rays may be explained by supposing the original agitation to be more or less violent, so as to cause a greater or less displacement of the particles of the medium from their positions of rest.

(3). The transverse agitations mentioned in (1), though they may of themselves be insufficient to produce any sensation of the phenomena of vision, still continue to be generated by the propagated motion of the ray from every point of its rectilinear course; and as this motion is continually supplied from the motion communicated by the original impulse; that in the direction of the ray must be continually diminished as it recedes from the origin. To determine the law of this, let us suppose the agitating cause to act equally in every direction from one point, so as to propagate a spherical wave of which this point is the centre. Now the *breadth* of this wave, or the thickness of the spherical shell containing *all* the particles which are in motion at any time, remains the same, while the radius (r) of the shell continually increases by the uniform propagation of

the motion; and hence the number of particles in motion at any time must be as the surface of the shell, or as r^2; and as the *vis viva*, distributed over all the surface is always the same, it must, for a given extent of surface, be inversely as the whole surface, or as $\frac{1}{r^2}$.

(4). To explain the different colours, we must again refer to the agitating cause. A single agitation produces only a single undulation, the progress of which is defined by the equation

$$v = F(x - a\,t) + f(x + a\,t),$$

which, in the corpuscular theory, would correspond to a single particle of light; but a succession of similar agitations of greater or less frequency is supposed to represent the phenomena of rays of different colours.

(5). The equation defining the motion of propagation of a wave is satisfied simultaneously by any number of different functions which express the different modes of agitation which produced the several rays; whence it is inferred, that all the undulations, whatever be the origin of each, may co-exist; that is, the several rays proceeding from different objects may cross each other in any manner, as, for instance, at the pupil of the eye, while the image of each is distinctly represented on the retina.

(6). We have seen (Problem 21) that each undulation produced by an oscillatory motion was composed of two semi-undulations, which occasioned in the particle of the fluid velocities exactly equal in their intensity, though opposite in the direction of the motions. Let us at first suppose that two whole undulations, moving in the same line, and in the same direction, differ half an undulation in their progress: they will then be superinduced on each other through one half of their length. In the supposed case of the coincidence of one half of each of the undulations, the interference will only take place with re-

spect to the parts so coinciding : that is, to the latter half of the first undulation, and the preceding half of the second : and if these two semi-undulations are of equal intensity, since they tend to give, to the same points of the ether, impulses directly opposite, they will wholly neutralise each other, and the motion will be destroyed in this part of the fluid, while it will subsist without alteration in the two other halves of the undulations. In such a case, therefore, half of the motion only would be destroyed.

If now, we suppose, that each of these undulations, differing in their progress by half the whole length of each, is preceded and followed by a great number of other similar undulations; then, instead of the interference of two detached undulations, we must consider the interference of two systems of waves, which may be supposed equal in their number and intensity. Since, by the hypothesis, they differ half an undulation in their progress, the semi-undulations of the one, which tend to cause in the particles of ether a motion in one direction, coincide with the semi-undulations of the other, which urge them in the opposite direction, and these two forces hold each other in equilibrium, so that the motion is wholly destroyed in the whole extent of these two systems of waves, except the two extreme semi-undulations, which escape from the interference. But these semi-undulations will always constitute a very small part of the whole series to be considered.

This reasoning is obviously applicable to such systems only as are composed of undulations of the same length : for if the waves were longer one than the other, however small their difference might be, it would happen at last that their relative position would not be the same throughout the extent of the groups ; and while the first destroyed each other almost completely, the following ones would be less in opposition, and would ultimately agree completely with each other : hence there would arise a succession of weak and strong vibrations analogous to the beatings which are produced by the coincidence of

two sounds differing but little from each other in their tone ; but these alternations of weaker and stronger light, succeeding each other with prodigious rapidity, would produce in the eye a continuous sensation only.

It is very probable, that the influence of a single luminous semi-undulation, or even of an entire undulation, would be too weak to agitate the particles of the optic nerve, as we find that a single undulation of sound is incapable of causing motion in a body susceptible of a sympathetic vibration. It is the succession of the impulse, which, by the accumulation of the single effects, at last causes the sonorous body to oscillate in a sensible manner ; in the same manner as the regular succession of the single efforts of a ringer is at last capable of raising the heaviest church bell into full swing. Applying this mechanical idea to vision, supported as it is by so many analogies, we may easily conceive that it is impossible for the two remaining semi-undulations, which have been mentioned, to produce any sensible effect on the retina ; and that the result of such a combination of the two systems must be the production of total darkness. If, again, we suppose the second system of undulations to be again retarded half an undulation more, so as to make the difference of the progress an entire undulation, the coincidence in the motions of the two groups will be again restored, and the velocities of oscillation will conspire and be augmented in the points of super-position ; the intensity of the light being then at its maximum.

Adding another semi-undulation to the difference in the progress of the two systems, so as to make it an interval and a half, it is obvious that the semi-undulations superinduced on each other, will now possess opposite qualities, as in the case of the half interval first supposed : and that all the undulations must in this manner be neutralised, except the extreme three semi-undulations on each side, which will be free from interference,

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Thus, almost the whole of the motion will again be destroyed, and the combination of the two pencils of light must produce darkness, as in the case first considered.

Continuing to increase the supposed difference by the length of a semi-undulation at each step, we shall have alternately complete darkness and a maximum of light, accordingly as the difference amounts to an odd or an even number of semi-undulations: that is, supposing always that the systems of undulations are of equal intensity: or if the one series were less vivid than the other, they would be incapable of destroying them altogether: the velocities of the one series would be subtracted from those of the other, since they would tend to move the particles of the ether in contrary directions, but the remainders would still constitute light, though feebler than that of the strongest single pencil. Thus the second pencil would still occasion a diminution of the light: but the diminution would be the less sensible as the pencil is supposed to be weaker.

It has been remarked, that when two systems of waves differ half an undulation in their progress, two of the semi-undulations must escape from interference; that six must be exempt when the difference amounts to three semi-undulations; and that, in general, the number of undulations exempt from interference is equal to the number of lengths of a semi-undulation separating the corresponding points of the two systems. While this number is very small in proportion to that of the waves contained in each system, the motion must be nearly destroyed, as in the case of the exemption of a single undulation. But it may be imagined that, as we increase the difference of the progress of the two pencils, the undulations exempted from interference may become a material portion of each group, and that it may finally become so great as to separate the groups entirely from each other; and in this case the phenomena of interference would no longer be observable. If,

for example, the groups of undulations consisted but of a thou-
sand each, a difference of one-twentieth of an inch in their
routes would be much more than sufficient to prevent the inter-
ference of rays of all kinds.

But there is another much more powerful reason which pre-
vents our perceiving the effects of the mutual influence of the
systems of waves when the difference of their routes is consider-
able; which is the impossibility of rendering the light sufficiently
homogeneous : for the most simple light that we can obtain con-
sists still of an infinity of heterogeneous rays, which have not
exactly the same length of undulation; and however slight the
difference may be, when it is repeated a great number of times,
it produces, of necessity, as we have already seen, an opposition
between the modes of interference of the various rays which
then compensates for the weakening of some by the strengthen-
ing of others; while the shades of colour are not sufficiently
distinct to allow the eye to remark the difference.   This is
without doubt the principal reason why the effects of the mutual
interference of the rays of light become insensible when the
difference of their routes is very considerable, so as to amount
to 50 or 60 times the length of an undulation.

It has already been laid down as one of the conditions ne-
cessary for the appearance of the phenomena of interferences
that the rays which are combined should have issued at first from
a common source : and it is easy to account for the necessity of
this condition by the theory which has now been explained.
Every system of waves, which meets another, always exercises
on it the same influence when their relative positions are the
same, whether it originates from the same source or from
different sources; for it is clear that the reasons, by which their
mutual influence has been explained, would be equally applica-
ble to either case.   But it is not sufficient that this influence
should exist, in order that it may become sensible to our eyes :

and for this purpose the effect must have a certain degree of permanence. Now this cannot happen when the two systems of waves which interfere are derived from separate sources. For it is obvious that the particles of luminous bodies, of which the vibrations agitate the ether, and produce light, must be liable to very frequent disturbances in their oscillations, in consequence of the rapid changes which are taking place around them, which may nevertheless be perfectly reconciled, as we have seen, with the regular continuance of a great number of oscillations in each of the series separated by these perturbations. This being admitted, it is impossible to suppose that these perturbations should take place simultaneously and in the same manner in the vibrations of separate and independent particles; so that it will happen, for example, that the motions of the one will be retarded by an entire semi-oscillation, while those of the other will be continued without interruption, or will be retarded by a complete oscillation, a change which will completely invert the whole effects of the interference of the two systems of undulations which originate from them; since if they had agreed on the first supposition, they would totally disagree on the second. Now these opposite effects, succeeding each other with extreme rapidity, will produce in the eye a continuous sensation only, which will be a mean between the more or less lively sensations that they excite, and will remain constant, whatever may be the difference of the routes described.

But the case is different when the two luminous pencils originate from a common source: for the two systems of waves, having originated from the same centre of vibration, undergoing these perturbations in the same manner and at the same instant, undergo no changes in their relative positions: so that if they disagreed in the first instance at any given point, they would continue to disagree at all other times; and if their motions co-operated at first, they would continue to agree as

long as the centre of vibration continued to be luminous: so that in this case, the effects must remain constant, and must therefore be sensible to the eye. This is therefore a general principle, applicable to all the effects produced by luminous undulations; that in order to become sensible, they must be permanent.

We have hitherto supposed that the two systems of waves are moving exactly in direction, and that consequently their elementary motions, to be combined with each other, were precisely limited to one single line: this is the simplest case of interference, and the only one in which the one motion can be completely destroyed by the other; for in order that this effect may be produced, not only the two forces must be equal and in contrary directions, but they must also act in the same right line, or be directly opposed to each other.

The phenomenon of coloured rings, and that of the colours of crystallised plates, present a particular case of interference, in which the undulations are exactly parallel.

But in the phenomena of diffraction, or in the experiment with the two mirrors (see **Dr. Young's Experiments, Philosophical Transactions, 1803, or Coddington's Optics, p. 154.**) the rays which interfere always form sensible though very small angles with each other. In these cases the impulses to be combined with each other at the same points, as belonging to the two systems of undulations, will also act in directions forming sensible angles with each other: but on account of the smallness of these angles, the result of the two impulses is almost exactly equal to their sum, when the impulses act in the same direction, and to their difference, when they are in contrary directions. Thus, in the points of agreement or disagreement, the intensity of the light will be the same as if the directions agreed more perfectly; at least the nicest eye will

not be able to discover any difference in them. But although, with respect to the intensity of the light, this case of interference resembles that which has already been considered, there are other differences which modify the phenomenon very greatly, both with respect to its general form, and to the circumstances necessary for producing it.

We may take, as a convenient example, the case of diverging rays originating from the same luminous point, and reflected by two mirrors slightly inclined to each other, so as to produce two pencils meeting each other in a sensible angle: the two systems of waves will then meet each other with a slight inclination; and it follows from this obliquity, that if a semi-undulation of the first system coincides perfectly in one point with a semi-undulation of the second, urging the fluid in the same direction, it must separate from it to the right and left of the point of intersection, and must coincide, a little further off, on one side with the preceding semi-undulation, which is in a contrary direction, and on the other side with the following semi-undulation, and then be separated from this again, and at a distance twice as great as the first, must coincide with the second semi-undulation before and behind it, of which the actions will coincide with its own: whence there will arise, on the surface of this undulation, a series of lines, at equal distances from each other, in which the motion is destroyed and doubled alternately by the action of the second series. Thus if we receive this luminous undulation on a white card, we shall observe on it a series of dark and bright stripes, if the light employed is homogeneous; or coloured fringes of different tints, if we employ white light for the experiment.

Fundamental Laws of the Reflexion and Refraction of Light on the undulatory Theory.

By a reasoning closely analogous to that used in Problem

21, it appears that the undulations of an elastic medium will on encountering an obstacle be either wholly or partially reflected; the law of its direction after reflection we shall presently proceed to determine. We may illustrate many of the particulars by the consideration of the laws of the impact of elastic balls; and thus be enabled to conceive how part of a ray may be reflected at the point of its immergence from a denser to a rarer medium. But it appears, from the difference of intensity in an incident and reflected ray, that a certain portion of it is transmitted or *absorbed* whatever may be the nature of the reflecting surface; whence, since the whole *vis viva* remains the same after impact as before, that which is apparently lost is converted into heat, or is employed in raising the temperature of the body, as appears from experiment.

We now deduce the law of the equality of the angles of incidence and reflection. (Fig. 5).

Let *ED* and *FG* be two incident rays, proceeding from the same centre of undulation, which we may suppose to be at an infinite distance, so that they may be parallel to each other: let *AB* be the reflecting surface; let *GI* be drawn perpendicular to *FG* and *ED*: then *GI* will represent the surface of the wave at the moment when it meets the reflecting surface in *G*. Now, according to the principle of Huygens, we may consider each of the points agitated by the wave, as *G* and *D*, in the light of new centres of agitation acting independently, and uniting rays in an infinity of directions, and with different intensities. It would, no doubt, be difficult to determine the law of the variation of their intensities in different directions about these points: but happily there is no occasion for such a determination; for, whatever the law might be, it is evident, that the rays passing from two points in parallel directions would be similarly affected by it, and must possess the same intensity and the same elementary direction of oscillation: so that this principle is

sufficient to enable us to judge of the direction in which the resulting undulations can be propagated. In fact, we may consider the reflected undulation at a distance from AB infinitely great, in comparison with GD, and other intervals of the same order: and supposing GK and DL to be two elementary rays that have been reflected, and that are proceeding to contribute to the formation of an elementary point of this distant undulation, and therefore parallel; and the angle KGB being equal to EDA; it is clear that the elementary motions transmitted in the lines GK and DL will agree perfectly with each other: for on account of the equality of the angles, if we draw DC perpendicular to GK, the two triangles GDC and DIG will be equal, and consequently GC will be equal to DI. But DI is the portion of the path of the incident ray ED, which it has described in its passage to the surface, after the description of EG by its collateral ray, and GC is the portion of the path of the ray reflected at G, which it has to describe beyond that which is reflected at D, in order to arrive at the point of their meeting: consequently when they meet they will both have described the same space, and will perform their motions in perfect agreement. But this would no longer be true, if the direction of the reflected rays were Gk and Dl, which are supposed also to meet in a point infinitely distant, but not to make an angle with the surface equal to EDA; for then the interval Gc, comprehended between the point G and the end of the perpendicular Dc, being no longer equal to ID, the paths described by the rays in order that they may arrive at the point of meeting, are no longer equal, and their oscillations at this point must be more or less discordant; now we may always take G at such a distance from D that the difference of GC and ID may be equal to a semi-undulation ; which will establish a complete discordance at the point of concourse, between the oscillations reflected along the lines Gk and Dl; and as they are besides of equal intensities, they will mutually destroy each

other; and there can consequently be no light reflected in that direction.

So true is it that the elementary ray *Dl* is neutralised in this case, by that which comes from the point *G*, that if we suppress this last, and the other rays which are sufficiently near to co-operate with it in counteracting *Dl*, we give, or rather we restore to the latter, the faculty of appearing in its place. The different elementary rays, reflected at *D*, are so much the more capable of diverging, as the extent of the reflective surface is the more confined on each side of this point; for the elementary ray *G′ k′* proceeding from a point *G′* situated at the same distance from *D* as the point *G*, counteracts the oscillations of *D l* at the point of meeting, as well as the ray *G k*; and the general mode of representing these mutual destructions of the elementary rays is to consider each intermediate ray *D l* as destroyed by the half, in intensity, of the ray *G k*, together with half of *G′ k′*: and then the remaining halves of these two rays by the halves of the next on each side, and so forth.

If we divide in this manner the surface of the mirror into a series of parts *DG′*, *G′G″*, &c. equal to *GD*; the elementary rays reflected at the points *G*, *D*, *G′*, *G″*, all directed to the same point of concourse at an infinite distance, and consequently parallel to each other, will differ by pairs half an undulation in their route: thus, for example, the ray *G k* will be found half an undulation in advance of the ray *D l*, and this the same distance in advance of the ray *G′ k′*, and so forth: for the same reason, the ray proceeding from the middle of the line *GD* will be completely at variance with the ray from the middle of *DG′*, and a similar discordance will take place between the rays reflected from all the other corresponding points of *GD* and *DG′*: in the same manner, all the rays reflected at the different points of *DG′* will be completely at variance with these which are reflected at the corresponding points of *G′G″*, and so forth: now the

intervals *GD, DG', G'G"*, &c. being equal to each other, the quantity of rays which they reflect is the same: we may, therefore consider each pencil of elementary rays reflected in this direction by an interval, *DG'*, as destroyed by the half, in intensity, of the rays of the preceding pencil, and by the half of the following pencil. If the surface is limited, and includes an even number of these intervals, the two remaining halves of the extreme pencils will be completely at variance when they meet, and will destroy each other at this point, so that no reflected light will be visible in this direction; but if the number of intervals is odd, the light reflected in this direction will be as little extinguished as possible, the remaining halves of the extreme pencils remaining in perfect agreement with each other. It must, however, be remarked, that in this case, the light diffracted, in the direction *G k*, will be much weaker than that which has been reflected in the direction *GK*, since all the rays proceeding from the surface, and uniting in the point of concourse, have described equal routes, and co-operate in their effects. All the consequences of this theory are confirmed by experiment. To give an idea of the extreme rapidity with which the light must be diminished in proportion as the direction *GK* deviates from that of the regular reflection, it may be added, that even when we can reckon on the surface of the mirror only five intervals such as *GD*, which give differences of half an undulation between their extreme rays, the intensity of the light reflected in the direction *G k* is only, according to this theory, $\frac{1}{8.0}$ of that of the light regularly reflected; and when the mirror is of a moderate breadth, it may be easily understood how very near the direction of *G k* must be that of *GK*, in order that it may contain but five intervals such as *GD*, or that there may be but five semi-undulations difference in the paths of the rays proceeding from the two extremities of the mirror.

It is easy to verify the consequences of this theory by

throwing, in a darkened room, the rays proceeding from a luminous point on a metallic mirror, or a glass blackened at the back, of which the upper surface is covered with a coat of very opaque black, with the exception of a long and very narrow surface, comprehended between two right lines, which make a very acute angle with each other, so that the breadth of the reflecting space continually diminishes as it approaches the angular point. If we place the mirror at a sufficient distance, and receive the reflected light on a white card, and then examine it with a magnifier, we shall remark that the pencil reflected by the part near the angle, is much broader than that which comes from the remoter part; and that consequently the divergence of the reflected rays is so much the greater, as the reflecting space is narrower. This manner of considering the nature of reflection, not only explains why the rays are not subjected in their progress, to the ordinary law of the equality of the angles of incidence and reflexion, when the surface is narrowed or discontinuous, but it even furnishes the means of computing their comparative intensities in their new directions. It has also the advantage of giving a clear and precise idea of that which constitutes a specular polish.

We must not consider the surface of the best polished mirror as a perfect plane : it is evident on the contrary, as Newton has already remarked, even from the mechanical process of polishing, that it must be roughened by an infinity of little projections ; for the fine powder, which is employed for this purpose, can only scratch it in every direction, and it is only the extreme fineness of these scratches that renders them imperceptible. But what is the degree of fineness that they must possess, in order to produce a regular reflexion ? This may easily be inferred from the explanation that has been given of the ordinary law of reflexion. For if the points G and G'' instead of being situated exactly in the mathematical plane ADB, are a little

above or below this plane, there will arise, in the paths of the
rays $G'k$ and $G''k'$, a small difference, which will lessen their
total discordance with the ray $D\,l$; and in the particular case of
a perpendicular incidence, for example, this difference would be
twice the projection of the points G and G'' above the plane
ADB: if, therefore, this difference were the hundredth part of
the breadth of a luminous undulation, the difference in the routes
which it would occasion, would be the fiftieth part of an undu-
lation : now so small an alteration from the complete discordance
of the elementary rays, would not produce any sensible light in
the direction $D\,l$, as the calculation founded on the law of inter-
ferences demonstrates. It is, therefore, sufficient that the pro-
jection of the elevations, or the depth of the depressions, should
be very small in proportion to the magnitude of an undulation
of light, in order that the surface of the mirror should reflect no
sensible light, except at an angle equal to the angle of inci-
dence; and when the greatest inequalities do not exceed the
hundredth of an undulation, for example, that is, the four or
five millionth of an inch for the yellow rays, the mirror must
necessarily exhibit a perfect polish; and there is reason to sus-
pect that, when the scratches are tolerably regular, it will have
the appearance of a bright polish, even if they are incomparably
larger than this.

There is a consequence of this explanation which requires to
be remarked. Since the length of the undulations are different
for the different kinds of coloured rays which constitute white
light, it may be imagined that the asperities of the surface may
be of such a magnitude as to afford a pretty regular reflexion of
the longest, those of the red rays, and yet to dissipate a con-
siderable portion of the violet rays, the length of undulation of
which is shorter by a third; so that in the regularly reflected
image of a white object, the red and orange rays may predomi-
nate, while the green, and especially the blue and violet rays,

may be in a smaller proportion, so that the tint would become reddish: and this result is confirmed by experiment. Instead of carrying the polish to a certain point only, which it might be difficult to ascertain, employ a mirror merely ground plane, and worked with fine emery, and incline it to the incident rays until you begin to distinguish a sufficiently distinct image of a white object, seen by reflexion; this image will appear tawny, and even of an orange red colour, like that of the setting sun, if the object is so bright as to be visible without too much inclination of the mirror. The tint of the image is the same, whatever the nature of the reflecting substance may be, whether of steel, for example, or of a greenish *crown glass*. In proportion as the obliquity of the mirror increases, the image becomes brighter and more brilliant, and when it becomes nearly parallel to the incident rays, the reflexion is as regular and almost as abundant as if it had been perfectly polished. We see, in this experiment, that the obliquity of the mirror produces the same effect as a diminution of its roughness would do: and it is easy to see the reason of this; for the roughnesses only alter the regularity of the reflection in proportion to the differences of the routes which they occasion: and it is easy to demonstrate geometrically that these differences become the smaller as the obliquity of the rays is greater.

We may now apply to the laws of refraction the same considerations which have served us for explaining the laws of reflexion. Let AB (fig. 6) be the surface separating two mediums in which light is not propagated with the same velocity. We may still suppose the incident rays FG and ED to have proceeded from a point infinitely distant, and consequently to be parallel to each other: and we may investigate the effects produced by the elementary rays refracted at an infinitely great distance only, in comparison with the interval GD, or other quantities of the same order, in order to simplify the state of the problem. Drawing GI perpendicular to the incident rays,

this line will show the direction of the undulation, or, in other words, the corresponding motion of the undulation will arrive simultaneously at G and I, and ID is the additional space that ED has to describe, in order to arrive at the surface. In the same manner, if we consider two elementary refracted rays, proceeding from the points G and D, and tending to a point infinitely distant, in the directions GK and DL, and suppose DM to be perpendicular to them, DM will be the portion of the path that the ray GK must describe, beyond its companion, reckoning from the surface, in order to arrive at the point of meeting. These rays will, therefore, arrive at this point in equal times, if the light describes the distance GM in the same time as ID. Now it is clear, that this only can happen if the two spaces are proportional to the lengths of the undulations of light in the respective mediums; or, representing the lengths of the undulations by d and d', if we have $GM : DI = d' : d$.

But, taking GD for the radius, GM will be the sine of the angle IGD: but IGD is equal to the angle of incidence IDP, and GDM to the angle of refraction QDL; consequently the sine of the angle of refraction must be to that of the angle of incidence as d' to d, in order that the two elementary refracted rays, which we are considering, may perfectly agree at their point of meeting : and this condition being equally fulfilled by all other elementary rays proceeding from the different points of the surface AB, which are re-united at the same point, all their undulations will perfectly coincide in this point, and will co-operate in their effect. It would be otherwise with the elementary rays Gk and Dl, tending also to a remote point, but in a different direction; for then Gm, being greater or smaller than GM, is not described in the same interval of time as ID, and one of these rays must necessarily be in advance of the other; now G may always be taken at such a distance from D, that this difference in their paths may be precisely equal to half

'the length of an undulation : so that for every elementary ray
$D\,l$, which departs from the direction DL, there is always
another ray $G\,k$ directed towards the same point, which differs
from it in the length of its route, by half an undulation ; and
whatever may be the law of variation in the intensities of the
elementary rays which would originate in the agitations at G
and at D, as proceeding in different directions, when separately
considered, it is clear that, the circumstances being exactly
similar for the two series of vibrations which are propagated in
the parallel directions, $D\,l$ and $G\,k$, the intensities of these will
always be the same, as well as the directions of the elementary
motions ; and since these undulations differ in their progress by
half an undulation, their motions will neutralise each other :
and it will be impossible that any luminous oscillation should
become sensible in the second medium, except in that direction
which makes such an angle of refraction, that its sine shall be
to the sine of the angle of incidence as d' is to d.

In the case of the divergence thus neutralised, it is not only
the oscillatory motions that are destroyed, but also the conden-
sations and dilatations which accompany them : in short, every
thing being symmetrical and equal among the quantities with
opposite signs that belong to the primitive motions, the same
must be true of the elementary undulations which are derived
from them : and this equality is sufficient to cause the mutual
destruction of all the quantities with opposite signs that are
concerned, whether velocities direct and retrograde, or con-
densations and dilatations, when any one of the positive quantities
is neutralised by a corresponding negative quantity ; or when
there is a difference of half an undulation in the lengths of the
paths described.

It may here be remarked again, as in the case of reflexion,
that when the surface AB is not infinite, some elementary rays
are always emitted by its extremities, which are not totally

destroyed, unless the intervals, like *GD,* answering to the difference of a semi-undulation, that are contained in *DL,* happen to be even in number with the extent of the surface. But when this breadth is at all considerable, the diffracted light, that spreads from the edges, is much fainter than that which has been regularly refracted. For further details of this theory of refraction, the reader is referred to the Notes on the Memoir in the Collection of the *Savans Etrangers.*

When the velocity of the propagation of light remains the same in all directions, for the same medium, the relation of *d* to *d',* and consequently that of the sines of the angles of incidence and refraction, remains constant, and the light follows the well known law of ordinary refraction. But there are substances in which the velocity of propagation varies in the same medium, with the direction of the rays, and then those which are thus affected are no longer refracted in the same manner.

The relation which we have just found, between the sines of the angles of incidence and of refraction, agrees perfectly with the experiment of Mr. Arago, which shows that the lengths of the undulations of light in different media are to each other as the sines of incidence and of refraction at the passage of the light from one medium into the other: and this relation explains at the same time, why the plates of air and of water, which reflect the rings of the same colours, are to each other as the sines of incidence and refraction of the light which passes from air into water.

If we generalise the considerations which have been employed for explaining the ordinary law of refraction in the particular case of a continued and extensive surface, we may determine, by means of the same formulas which represent the phenomena of diffraction, the much more complicated laws which are followed by the refracted rays, when the surface is

narrow or interrupted; and we always obtain, in this manner, results which are conformable to experience; which proves both the accuracy and the generality of the principles of Huygens, and of that of interferences, on which the whole of this theory is founded.

It is impossible to conclude this concise account of refraction, without offering some theoretical views of an optical phenomenon which always accompanies it, which has been much studied, but which is, perhaps, still very little understood; that is, the division which light undergoes in passing through a prism, and to which the name of dispersion has been given, because it separates, and in some measure disperses the coloured rays, of which white light is composed, and makes them follow different routes. It results from this phenomenon, that the rays of different colours are not all equally refracted, or that the undulations of different lengths are not propagated with the same velocity in the same mediums: for it follows, from the explanation which has been given of refraction, that the relation between the sines of incidence and refraction, for each kind of undulations, must always be the same with the relation of the velocities of propagation in the two mediums; so that if the different rays passed through them with the same velocities they would be equally refracted, and there would be no dispersion. We must therefore suppose that, in refractive mediums, the undulations of different lengths are not propagated with the same velocity, or, in other terms, are not shortened in the same proportion. This consequence appears, at first sight, to be contradictory to the results of the elaborate calculations of Mr. Poisson on the propagations of sonorous undulations in elastic fluids of different densities: but it must be observed that, in this theory, the general equations are founded on the supposition, that each infinitely thin stratum of the fluid is repelled by the stratum in contact with it only, and thus that the accelerative

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force extends to infinitely small distances only, in comparison with the length of an undulation. This supposition is, without doubt, perfectly admissible for the undulations of sound, the shortest of which are some tenths of an inch in length : but it may possibly be inaccurate for the undulations of light, the longest of which are several thousand times shorter. It is very possible that the sphere of activity of the accelerative force, which determines the velocity of propagation of light in a refractive medium, or the mutual dependence of the particles of which it is composed, may extend to distances which are not infinitely small in comparison with the fifty thousandth of an inch: for such a supposition is not contradicted by anything that we know of their limited sphere of activity. Now it is easy to infer, from mechanical considerations, that if the sphere of activity of the accelerative forces extends actually to sensible distances, in comparison with the length of the luminous undulations, those which are the longest must be less retarded in their progress by the density of the medium, or less shortened, in proportion, than the shorter undulations, and must, consequently be less refracted : a conclusion which agrees with the only general rule, that has hitherto been experimentally discovered, for the phenomenon of dispersion.

However this may be, the facts sufficiently shew that luminous undulations of different lengths are actually transmitted with different velocities in the same refractive mediums, and in variable proportions, of which the laws are yet entirely unknown, but which seem to be very intimately related to the chemical nature of the substances. But are we to infer that there is any difference in the velocities of the different rays of light in the pure ether, which occupies the celestial space? To this question it is difficult to reply with any certainty; but the astronomical observations of Mr. Arago appear to give us a negative answer.

25. If any number of rays be incident parallel to the axis on the surface of an elliptic paraboloid, the reflected rays will all pass through each of two parabolas lying in the principal planes of the paraboloid.

We shall first exbibit the general analytical relation between the directions of an incident and reflected ray, whatever be the nature of the reflecting surface.

Let ρ, l, ρ', l denote respectively the angles made by an incident ray ρ and a reflected ray ρ' with any fixed line l, I the angle of incidence, and n, l the angle made by the perpendicular with l.

By taking a portion equal to unity on each of the lines ρ, ρ' reckoned from the point of reflexion and employing the same geometrical reasoning as if these two portions represented two equal forces whose resultant made an angle n, l with l, we should get, by projecting them on l, the equation

$$\cos \rho, l + \cos \rho', l = 2 \cos I \cdot \cos n, l,$$

which is the analytical expression of the law of reflexion, and consequently comprises the whole theory of Catoptrics.

If we refer the position of the point of reflexion, and the directions of the incident and reflected rays to three rectangular axes of co-ordinates, this equation may be replaced by the three following:

$$\cos \rho, x + \cos \rho', x = 2 \cos I \cdot \cos n, x,$$
$$\cos \rho, y + \cos \rho', y = 2 \cos I \cdot \cos n, y, \quad (1)$$
$$\cos \rho, z + \cos \rho', z = 2 \cos I \cdot \cos n, z;$$

n, x, n, y, n, z denoting the angles made by the normal at the point of reflexion with the axes of x, y, z.

In the example proposed, let the axis of the paraboloid be the axis of z, so that its equation may be

$$4\,a\,b\,z' = a\,y'^2 + b\,x'^2;$$

and the co-ordinates of any point of reflexion being x', y', z', the equations of the incident ray (parallel to the axis) will be

$$\left.\begin{array}{l} x - x' = 0 \\ y - y' = 0 \end{array}\right\},$$

and those of the reflected ray

$$\left.\begin{array}{l} x - x' = \mu\,(z - z') \\ y - y' = \nu\,(z - z') \end{array}\right\},$$

μ, ν being functions of x', y', z' to be determined; whence

$$\cos\rho',\ x = \frac{\mu}{\sqrt{1 + \mu^2 + \nu^2}},$$

$$\cos\rho',\ y = \frac{\nu}{\sqrt{1 + \mu^2 + \nu^2}},$$

Which values being substituted in the two first of equations (1) in which $\cos\rho,\ x = 0,\ \cos\rho, y = 0,\ \cos\rho, z = 1$,

$$\cos I = -\cos n,\ z = \frac{-1}{\sqrt{1 + p^2 + q^2}},$$

$$\cos n,\ y = -\frac{q}{\sqrt{1 + p^2 + q^2}},$$

$$\cos n,\ x = -\frac{p}{\sqrt{1 + p^2 + q^2}},$$

give

$$\mu = \frac{2\,p}{1 - p^2 - q^2},$$

$$\nu = \frac{q}{p} \cdot \mu = \frac{2\,q}{1 - p^2 - q^2},$$

and the equations of the reflected ray thus become

$$x - x' = -\frac{2\,p}{1 - p^2 - q^2}\,(z - z'), \qquad (2)$$

$$y - y' = - \frac{2\,q}{1 - p^2 - q^2}(z - z'),$$

whence $y - y' = \frac{q}{p}(x - x'),$

or, since, from the equation

$$4\,a\,b\,z' = a\,y'^2 + b\,x'^2,$$

$$p = \frac{x'}{2\,a}, \text{ and } q = \frac{y'}{2\,b}, \text{ and } \therefore \frac{q}{p} = \frac{a\,y'}{b\,x'},$$

$$y - y' = \frac{a\,y'}{b\,x'}(x - x'),$$

or $\quad a\,y'\,x - b\,x'\,y + (b - a)\,x'\,y' = 0,$ (3)

which equation will represent successively any number of
straight lines, by giving the same number of different values to
x' and y' : so that if this equation be conceived to be the result
of the integration of a differential equation in which the variables
were x and y ; x' and y' would evidently be the arbitrary con-
stants introduced by the integration ; and a particular solution
obtained by the elimination of these would accordingly lead to
the equations of the curve through which all rays parallel to the
axis would pass after reflection.

Thus, by differentiating equations (3) with respect to x' and
y', and making the co-efficients of dx' and dy' equal to nothing
(because from the nature of the surface, the variation, of x' and y'
are independent) we arrive at the equations

$$a\,x + (b - a)\,x' = 0,$$

$$b\,y - (b - a)\,y' = 0 ;$$

from which $\quad x' = - \frac{a\,x}{b - a}, \text{ and } y' = \frac{b\,y}{b - a} ;$

$$\therefore \ x - x' = \frac{b\,x}{b - a}, \quad y - y' = \frac{b\,y}{b - a},$$

and
$$\frac{2p}{1 - p^2 - q^2} = -\frac{4(b-a)x}{4(b-a)^2 - x^2 - y^2},$$

$$\frac{2q}{1 - p^2 - q^2} = \frac{4(b-a)y}{4(b-a)^2 - x^2 - y^2};$$

by the substitution of which expressions, the equations (2) become

$$x^2 = 4(b-a)(b-z)$$
$$y^2 = 4(b-a)(z-a)$$

which represent two parabolas in the planes of xz and yz through which all the rays pass.

<div align="right">Q. E. D.</div>

It may be observed that the principal parameter of each of these parabolas is the more diminished as b becomes more nearly equal to a: or the more nearly the paraboloid approximates to a surface of revolution; and that the point where each ray passes through each parabola approaches more nearly to its vertex.

In the limiting case when $b = a$, the two last equations give $x = 0$, and $y = 0$; and since the vanishing ratio of x and y is in this case a ratio of equality, the combination of these two equations gives $\frac{a-z}{z-a} = 1$, or $z = a$; which shows that all the reflected rays pass through the focus: which is a well-known property of the paraboloid of revolution.
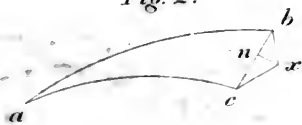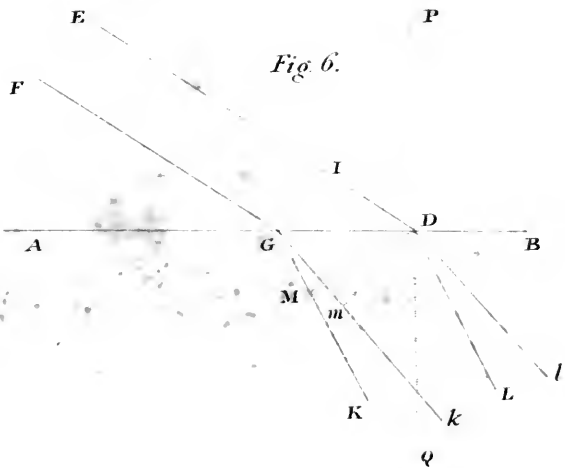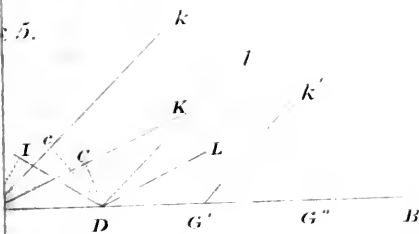
Fig. 2.

Fig. 6.

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